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Algorithms for Frames and Lineality Spaces of Cones*

Roger J.-B. Wets and Christoph Witzgall****

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(November 3, 1966)

A frame of a cone C is a minimal set of generators, and the lineality space L of C is the greatest linear subspace contained in C. Algorithms are described for determining a frame and the lineality space of a cone $C(S)$ spanned by a finite set S. These algorithms can be used for determining the vertices, edges, and other faces of low dimension of the convex hull of a finite set $H(S)$. All algorithms are based on the simplex method of linear programming. The problem of finding the lineality space can be successively reduced to problems in spaces of lower dimensions.

Key Words: Algorithm, cone, convex hull, face, frame, lineality space, linear programming.

Introduction

Let

$$S=\{A_1, \ldots, A_n\}$$

be a family of points in R^m. We denote by $L(S)$ its *linear hull*, by $H(S)$ its *convex hull*, and by $C(S)$ its *conical* or *positive hull*, i.e., the convex polyhedral cone expressible as the set of all nonnegative weighted sums of elements of S. We use the conventions $L(\phi)=C(\phi)=\{0\}$.

A subfamily $T \subseteq S$ is called a *frame* [4][1] of $C(S)$ if $C(T)=C(S)$ but $C(T-\{A_j\}) \neq C(T)$ for each $A_j \in T$. The greatest linear subspace contained in $C(S)$ is called its *lineality space* [5]. Two main problems are, given S, to find a frame of $C(S)$ and to determine the lineality space of $C(S)$. These two problems are closely related.

Several important problems are equivalent to or included in these two main problems. Consider, for instance, the system of linear inequalities

$$A_j^T X \leq b_j, \qquad j=1, \ldots, n,$$

for $X \in R^m$. Removing redundant constraints amounts to finding a frame for the cone spanned by the vectors $\hat{A}_j=\begin{pmatrix} A_j \\ b_j \end{pmatrix}$, $j=1, \ldots, n$, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Determining the dimension of a polyhedron given by linear inequalities, can be reduced to determining the dimension of a polyhedral cone $C=\{U \mid A^T U \leq 0\}$ (see [10]). As to the latter problem, it suffices to note that the linear hull $L(C)$ is the orthogonal complement of the lineality space of the polar cone $C^p=C\{A_1, \ldots, A_n\}$, where $A_1, \ldots, A_n$ are the columns of A. This problem also arises if the uniqueness of an optimal solution to a linear program is to be established in the presence of degeneracy.

Clearly, the problem of finding the vertices of a convex hull $H(S)$ can be solved by finding the frame of a suitable cone. It should not be confused with the problem of finding the vertices of a polyhedron defined by linear inequalities. The latter problem corresponds to finding the *facets*, that is, the proper faces of highest dimension, of some $H(S)$. Even a moderate number of inequalities is apt to generate a huge number of vertices [6, 9], and by the same token, the number of facets of $H(S)$ may be extremely large compared to the cardinality of S. Thus finding the facets of $H(S)$ is inherently more difficult than finding the vertices of $H(S)$.

The problem of finding the *edges* of $H(S)$, and other facets of low dimension, may still be expected to be essentially easier than to find the facets of $H(S)$. Indeed, one is tempted to conjecture that an efficient determination of the facets requires prior determination of the lower dimensional faces.

Note that finding the edges of $H(S)$ could be accomplished by finding the vertices of $H(S_2)$, where

$$S_2 := \{A_{(ij)}=\tfrac{1}{2}(A_i+A_j) \mid i<j\}.^2$$

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**Mathematics Research Laboratory, Boeing Scientific Research Laboratories, Seattle, Wash. 98110.

[1] Figures in brackets indicate the literature references at the end of this paper.

[2] The symbol ":=" stands for "is defined by."

However, this procedure is not recommended since the cardinality of S_2 tends to become quite large. We shall prefer a more compact technique, which works with S rather than S_2 or its analogs for higher dimensional faces, and which is closely related to an algorithm for solving the second main problem of determining the lineality space of a cone. The faces of $H(S)$ will be characterized in terms of sign patterns of matrices representing $H(S)$. This aspect is examined more closely in [13].

Charnes [2], Motzkin, Raiffa, Thompson, and Thrall [11], and Farrell and Fieldhouse [14] have proposed methods for finding the facets *and* the vertices of a convex polyhedron P determined by a system of linear inequalities. On the other hand, Goldstein [7] suggests an algorithm for solving the same problem where P is given as the convex hull of a finite set of points. Recently, Thompson, Tonge, and Zionts [12] published a method for removing redundant constraints. One of our algorithms for determining a frame of the cone $C(S)$ essentially coincides with theirs.

We shall give algorithms both for finding a frame for $C(S)$, and for determining the lineality space of $C(S)$. As will be pointed out, the latter algorithm can be modified to decide whether a given subset of S characterizes a face of $C(S)$. Applied to single elements of S, such an algorithm can also be used to find the frame of a pointed cone $C(S)$ and, therefore, the vertices of a convex hull $H(S)$.

All algorithms in this paper are based on the simplex method for linear programming (see, for instance, [3]). Not surprisingly, there are "primal" and "dual" algorithms in each case. We shall describe a primal algorithm for finding a frame and a dual algorithm for determining the lineality space of $C(S)$. The remaining two algorithms are readily constructed by the reader.

It has been our experience with the above algorithms that in the presence of degeneracies round-off errors may cause cycling if zero tolerances are not chosen and handled correctly. This danger is diminished if degeneracy provisions are made, but we shall not include such provisions in the descriptions of our algorithms.

1. General Remarks

A set S may contain more than one frame for $C(S)$, and the cardinalities of these frames may differ. For instance, if $C(S)$ is a linear subspace of dimension l, then the cardinality of a frame T may range from $l+1$ to $2l$. We shall only concern ourselves with finding *some* frame for $C(S)$, and not, for instance, a frame of minimum cardinality.

A point $A_c \in S$ is *redundant in* S if $C(S-\{A_c\}) = C(S)$; otherwise, A_c is *necessary in* S. The algorithms for finding a frame will consist of repeated applications of a subalgorithm which determines for some $A_c \in \bar{S} \subseteq S$ whether it is redundant or necessary in $\bar{S}$. The subset $\bar{S}$ arises from S by deleting those columns that were already found to be redundant.

Now let L be the lineality space of $C(S)$. Clearly,

$$L = C(S) \cap C(-S).$$

We partition the set S into two classes, its *lineal part* $S_L := S \cap L$ and its *conical part* $S_C := S - S_L$. Then

$$L = C(S_L). \tag{1.1}$$

PROOF. $S_L \subseteq L$, hence $L \supseteq C(S_L)$. To prove the other inclusion, suppose $X \in L$. Then $X \in C(S)$ and $X \in C(-S)$. Hence $X = \Sigma A_i u_i$, $u_i \geq 0$, and $X = \Sigma(-A_i)v_i$, $v_i \geq 0$. It follows that

$$\Sigma A_i (u_i + v_i) = 0, \qquad u_i + v_i \geq 0.$$

If $u_i + v_i > 0$, and in particular if $u_i > 0$, then clearly $-A_i \in C(S)$ and, therefore, $A_i \in S_L$. Thus $X \in C(S_L)$—[3].

Let $\bar{S}$ be a subset of S such that $\bar{S} \supseteq S_L$. Let $\bar{L}$ denote the lineality space of $C(\bar{S})$. Then (1.1) gives

$$L = C(S_L) \cap C(-S_L) \subseteq C(\bar{S}) \cap C(-\bar{S}) \subseteq C(S) \cap C(-S) = L,$$

whence $L = \bar{L}$. As a consequence

$$\bar{S}_L = S_L \text{ for } S_L \subseteq \bar{S} \subseteq S. \tag{1.2}$$

We shall describe an algorithm for finding the lineal part S_L of a given set S. Again, this algorithm will be based on a subalgorithm, which decides whether a specified $A_c \in \bar{S} \subseteq S$ is in $\bar{S}_L$. The set $\bar{S}$ arises from S by deleting those points A_i which were already found to be in S_C. This deletion is justified by (1.2).

We proceed to show how the problem of finding faces of $C(S)$ relates to the problem of determining lineality spaces. The faces of a cone C are *extreme subsets*, that is, if the sum of points in C lies on a face F of C, then so do all the summands. More precisely,

(1.3) *$X \in F$, and $X = \Sigma Y_i u_i$ with $Y_i \in C$ and $u_i \geq 0$, imply $u_i = 0$ for $Y_i \notin F$.*

This property characterizes faces. If F is a face of $C(S)$, and if $S_F := F \cap S$, then

(1.4) *$L(S_F)$ is the lineality space of $C(S \cup (-S_F))$.*

PROOF. $C(S \cup (-S_F)) \subseteq C(S_F \cup (-S_F)) = L(S_F)$. Now suppose

$$X \notin C(S \cup (-S_F)) \cap C((-S) \cup S_F).$$

Then there exist u_i, w_i, v_i, $z_i \geq 0$ such that

$$X = \sum_{A_i \in S} A_i u_i - \sum_{A_i \in S_F} A_i w_i = - \sum_{A_i \in S} A_i v_i + \sum_{A_i \in S_F} A_i z_i.$$

[3] The symbol "—" marks the end of a proof.

Hence

$$\sum_{A_i \in S} A_i(u_i + v_i) = \sum_{A_i \in S_F} A_i(w_i + z_i).$$

By (1.3), $u_i + v_i = 0$ if $A_i \notin S_F$. In particular, $u_i = 0$ if $A_i \notin S_F$. Hence

$$X = \sum_{A_i \in S_F} A_i u_i - \sum_{A_i \in S_F} A_i w_i,$$

which implies $X \in L(S_F)$. —

The points $A_1, \ldots, A_k$ (all in S) are said to *subdetermine* a face F of $C(S)$ if

$$F = L\{A_1, \ldots, A_k\} \cap C(S).$$

We then have

(1.5) *The points* $A_1, \ldots, A_k$ *in* S *subdetermine a face of* $C(S)$ *if and only if* $L\{A_1, \ldots, A_k\}$ *is the lineality space of* $C(S \cup \{-A_1, \ldots, -A_k\})$.

PROOF. Suppose $L\{A_1, \ldots, A_k\}$ is the lineality space of $C(S \cup \{-A_1, \ldots, -A_k\})$. In order to prove that $F := L\{A_1, \ldots, A_k\} \cap C(S)$ is a face, we have to show according to (1.3) that if $X \in F$, $X = \Sigma A_i u_i$ with $u_i \geq 0$, and $u_h > 0$, then $A_h \in F$. Suppose, therefore, that X and u_h are as above. Then

$$A_h = \frac{1}{u_h}(X + Z),$$

where

$$Z := -\sum_{i \neq h} A_i u_i \in C((-S) \cup \{A_1, \ldots, A_k\}).$$

Now X is in $L\{A_1, \ldots, A_k\}$, which is by hypothesis the lineality space of $C(S \cup \{-A_1, \ldots, -A_k\})$. Therefore, $X \in C((-S) \cup \{A_1, \ldots, A_k\})$, whence $A_h \in C((-S) \cup \{A_1, \ldots, A_k\})$. It follows that A_h is in the lineality space of $C(S \cup \{-A_1, \ldots, -A_k\})$. Thus $A_h \in L\{A_1, \ldots, A_k\} \cap C(S) = F$.

On the other hand, if $F := L\{A_1, \ldots, A_k\} \cap C(S)$ is a face, then $L(S_F) = L\{A_1, \ldots, A_k\}$ is the lineality space of $C(S \cup (-S_F)) = C(S \cup \{-A_1, \ldots, -A_k\})$ by virtue of (1.4). —

Note that if $A_1, \ldots, A_k$ subdetermine F, then $L(F) = L\{A_1, \ldots, A_k\}$. Hence

$$\dim (F) = k$$

if $A_1, \ldots, A_k$ are linearly independent. Thus any algorithm for determining lineal parts can be used for deciding whether a given set of points $\{A_1, \ldots, A_k\} \subseteq S$ subdetermines a k-face (= face of dimension k) of $C(S)$.

2. Cones and Equivalent Matrices

By definition, the point A_c is redundant in S if and only if $C(S - \{A_c\}) = C(S)$. This in turn is equivalent to $A_c \in C(S - \{A_c\})$, or

$$A_c = \sum_{j \neq c} A_j u_j, \qquad u_j \geq 0. \tag{2.1}$$

The point A_c is in the lineal part S_L of S, if and only if $A_c \in L = C(S) \cap C(-S)$. This is equivalent to $-A_c \in C(S - \{A_c\})$, or

$$-A_c = \sum_{j \neq c} A_j v_j, \qquad v_j \geq 0. \tag{2.2}$$

The standard "phase I" simplex procedures of linear programming are available for deciding the solvability of (2.1) or (2.2) (see, for instance, [3]). All that remains is to integrate such a procedure with the overall algorithm in a way that minimizes the rearrangement of data.

The points $A_1, \ldots, A_n$ of S can be represented by an $m \times n$ matrix

$$A := (A_1, \ldots, A_n).$$

The matrix A is in *canonical form* if it contains an $m \times m$ permutation matrix, called *basis*. Its columns are the *basic* columns; all other columns are *nonbasic*. If A is in canonical form, and if the column A_j is nonbasic, then the elements of A_j constitute the representation of the point A_j in terms of the basis of A. Criteria for redundancy and lineality will be therefore particularly simple if the matrix A is in canonical form.

Two matrices A and $\overline{A}$ are called *equivalent* if there exist (possibly rectangular) matrices T and $\overline{T}$ such that $TA = \overline{A}$ and $\overline{T}\overline{A} = A$. If and only if A and $\overline{A}$ are equivalent in this sense, then for all $X \in R^n$:

$$AX = 0 \text{ } \textit{if and only if} \text{ } \overline{A}X = 0. \tag{2.3}$$

Both (2.1) and (2.2) express the existence of a suitable *linear dependence*

$$AX = 0, \qquad X \neq 0$$

of the columns of A. By (2.3) equivalent matrices admit the same linear dependences. If, therefore, A is replaced by any equivalent matrix $\overline{A}$—preferably in canonical form—then the same columns (= columns with the same indices) will be redundant in the respective column sets, and the same columns will be in the lineal parts.

3. Determining a Frame of $C(S)$

For matrices A in canonical form the following criteria are immediate:

(3.1) *If* A_j *is nonbasic and* $A_j \geq 0$, *then* A_j *is redundant.*

(3.2) *If* A_j *is nonbasic and has exactly one positive entry* a_{rj}, *then the basic column* A_c *for which* $a_{rc} = 1$ *is redundant.*

The following two criteria hold whether A is canonical or not:

(3.3) *If the* i*th row of* A, *henceforth denoted* ${}_iA$, *contains exactly one negative entry* a_{is}, *then* A_s *is necessary.*

(3.4) *If* ${}_iA$ *contains exactly one positive entry* a_{is}, *then* A_s *is necessary.*

As was pointed out before, any "phase I" simplex procedure can be used for deciding whether a given column is redundant. We prefer a variant (described in [8]) without artificial variables which works on each infeasible row separately, treating it to some extent as an objective function while conserving the feasibilities already achieved. While this variant may not be the most efficient one for finding a first feasible solution—using a positive combination of the infeasible rows is in general better—the terminal situations of this variant are precisely the ones to which criteria (3.1) and (3.3) apply; it is therefore particularly easy to implement.

The matrix A will be repeatedly transformed and some of its columns may be deleted. In the following description of the algorithm, the symbol "A" will always refer to the particular matrix at hand. It will also be convenient, not to change the indexing of columns which remain after others have been deleted.

(3.5) *Primal algorithm for determining a frame:*

(i) (*Canonical form*) Use Jordan elimination for bringing A into canonical form. Every column is labeled "undecided."

(ii) (*Constant column*) Select a nonbasic undecided column A_c as "constant column." If there are no such columns, go to (viii); else proceed to (iii).

(iii) (*Pilot row*) If $A_c \geq 0$, delete A_c and return to (ii). Else select ${}_pA$ such that $a_{pc} < 0$; call it the "pilot row."

(iv) (*Pivot column*) If a_{pc} is the only negative entry in the pilot row, change the label of A_c from "undecided" to "necessary" and go to (ii). Else select a "pivot column A_l such that $a_{pl} < 0$ and $l \neq c$.

(v) (*Pivot row*) Select a "pivot row" ${}_rA$ such that

$$0 \leq \frac{a_{rc}}{a_{rl}} \leq \min \left\{ \frac{a_{ic}}{a_{il}} \middle| a_{ic} \geq 0, \quad a_{il} > 0 \right\}.$$

There exists always an index r of this kind, since either the minimum on the right-hand side is finite and assumed—we use the convention $\min(\phi) = +\infty$ —, or $r = p$ is a permissible choice.

(vi) (*Pivoting*) Execute a simplex step (= Jordan transformation) with a_{rl} as pivot. In the absence of degeneracies, this will increase the old entry a_{pc} while keeping nonnegative entries of A_c nonnegative.

(vii) (*Return*) If the new entry a_{pc} is still negative, keep the pth row as pilot row and go to (iv). Else go to (iii).

(viii) (*Termination*) If all basic columns are decided, terminate the procedure. Else select an undecided basic column A_c.

(ix) (*Clear basis*) Suppose $a_{rc} = 1$. If this is the only positive entry in ${}_rA$, then change the label of the column A_c from "undecided" to "necessary" and go to (viii). Else pivot so as to remove the undecided column A_c from the basis. Go to (iii) with A_c as constant column. (*End of the algorithm.*)

In the absence of degeneracies, this algorithm increases at every step the entry a_{pc} until it becomes nonnegative or remains the only negative entry in the pilot row. In the latter case, the column A_c has been decided. In the former case, a_{pc} stays nonnegative during all subsequent transformations, while some other negative element of A_c is being increased. Hence a decision on A_c will be reached after finitely many simplex steps.

The algorithm is sped up by checking after each iteration whether some of the columns or rows satisfy criteria (3.1), (3.2), (3.3), or (3.4).

A dual algorithm results if the subalgorithm which decides whether A_c is redundant or necessary is replaced by its dual. The primal decision algorithm tries to make a given column of A nonnegative (criterion (3.1)), and encounters criterion (3.3) if this is not possible. The dual decision algorithm aims at criterion (3.3), that is, it attempts to make all entries but one in some row of A nonnegative. If this attempt fails, then criterion (3.2) is encountered. This algorithm is a "phase I" procedure for the dual simplex method.

4. Determining the Lineality Space of $C(S)$

The following criterion is immediate for matrices in canonical form:

(4.1) *If* $A_j \leq 0$, *then* $A_j \in S_L$.

For any matrix A one has

(4.2) *If* ${}_iA \geq 0$, *then all columns* A_j *with* $a_{ij} > 0$ *belong to* S_C.

An algorithm analogous to the algorithm (3.5) can be based on these criteria. The pivot rules arise from those of the algorithm (3.5) by reversing the sign of the constant column. Whenever a nonnegative row ${}_iA$ is found, all columns A_j with $a_{ij} > 0$ are deleted.

Thus, ith row of the remaining matrix vanishes and is therefore deleted. Whenever a nonpositive column A_c is found, it is labeled "lineal," and this process is repeated until an empty matrix results or until all columns are labeled "lineal." There are primal and dual strategies available, the former aiming at nonpositive columns, the latter at nonnegative rows.

The algorithms are sped up if the following refinement of criterion (4.1) is used (for matrices in canonical form). (See [13]):

(4.3) *Suppose* $\mathrm{A}_c \leqq 0$, *and let* $\hat{\mathrm{A}}$ *be the matrix that arises from* A *if all rows* ${}_i\mathrm{A}$ *with* $\mathrm{a}_{ic} < 0$ *are deleted. Then* $\hat{\mathrm{A}}_j$ *is in the lineal (conical) part of* $\hat{\mathrm{A}}$ *if and only if* A_j *is in the lineal (conical) part of* A. *If* $\mathrm{A}_c < 0$, *then* $\mathrm{S}_C = \emptyset$.

PROOF. Not only A_c but also all basic columns A_h such that $a_{ih} = 1$ for $a_{ic} < 0$ are in S_L. These basic columns A_h span the linear subspace $E := \{X | x_i = 0 \text{ if } a_{ic} = 0\}$, which is therefore contained in the lineality space L of $C(S)$. We define $\tilde{A}_j$ by putting

$$\tilde{a}_{ij} := \begin{cases} a_{ij} \text{ if } a_{ic} = 0 \\ 0 \quad \text{if } a_{ic} < 0. \end{cases}$$

Then $\tilde{A}_j - A_j \in E \subseteq L$. Hence $\tilde{A}_j \in L$ if and only if $A_j \in L$.

Denote by $\tilde{S}$ the set of all $\tilde{A}_j$. It clearly suffices to show that $\tilde{A}_j \in \tilde{S}_L$ if and only if $A_j \in S_L$. Suppose $A_j \in S_L$. Then $\tilde{A}_j \in L$ and therefore $-\tilde{A}_j = \Sigma A_i u_i$, $u_i \geqq 0$. This relation remains true if we replace A_i by $\tilde{A}_i$: $-\tilde{A}_j = \Sigma \tilde{A}_i u_i$, whence $\tilde{A}_j \in \tilde{S}_L$. In the other direction, if $-\tilde{A}_j = \Sigma \tilde{A}_i u_i$, $u_i \geqq 0$, then

$$-A_j = \tilde{A}_j - A_j + \Sigma(\tilde{A}_i - A_i)u_i + \Sigma A_i u_i$$

but $\tilde{A}_j - A_j$, $\tilde{A}_i - A_i \in L \subseteq C(S)$. Hence $-A_j \in C(S)$ and, therefore, $A_j \in S_L$.

We proceed to describe a dual algorithm. The corresponding primal algorithm is readily constructed by the reader. Again it will be convenient to denote by A the particular matrix at hand, and not to change the indices of rows and columns if other rows and columns are deleted.

(4.4) *Dual algorithm for determining the lineality space of* C(S):

(i) (*Canonical form*) Use Jordan elimination for bringing A into canonical form. Label all zero columns of A "lineal."

(ii) (*Termination*) If A is empty or all columns of A are labeled "lineal," then terminate the procedure. Else proceed to (iii).

(iii) (*Objective row*) Select any row ${}_cA$ as "objective row."

(iv) (*Pilot column*) If ${}_cA \geqq 0$, then delete all columns A_j with $a_{cj} > 0$, delete ${}_cA$, and return to (ii). Else select a "pilot column" A_p with $a_{cp} < 0$.

(v) (*Pivot row*) If $A_p \leqq 0$, delete all rows ${}_iA$ with $a_{ip} < 0$, label all zero columns generated by this deletion (all columns, if $A_p < 0$) "lineal," and return to (ii). Else select a "pivot row" ${}_rA$ with $a_{rp} > 0$ and $r \neq c$.

(vi) (*Pivot column*) Select a "pivot column" A_l such that

$$0 \geqq \frac{a_{rl}}{a_{rl}} \geqq \max\left\{\frac{a_{cj}}{a_{rj}} \;\middle|\; a_{cj} \geqq 0, \quad a_{rj} < 0\right\}$$

(compare algorithm (3.5) step (v)).

(vii) (*Pivoting*) Pivot on a_{rl}. In the absence of degeneracies, this will increase the entry a_{cp} while keeping nonnegative entries of ${}_cA$ *nonnegative*.

(viii) (*Return*) If the new entry a_{cp} is still negative, keep the pth column as pilot column and go to (v). Else return to (iv). (*End of algorithm*)

In the absence of degeneracies, this algorithm will terminate in a finite number of steps (compare (3.5)). The algorithm is sped up by checking after each iteration whether some of the rows or columns satisfy criteria (4.1) or (4.2).

5. Determining k-Faces of $C(S)$

For finding all k-faces of $C(S)$ we employ a subalgorithm which decides for each set of k linearly independent elements of S whether or not they subdetermine a face. We do not raise the question of the most efficient arrangement of this subalgorithm within the overall search algorithm. Instead, we try to formulate the decision algorithm in a manner that does not preclude its implementation as a subalgorithm. To be more precise, in order to decide whether $A_1, \ldots, A_k$ subdetermine a k-face, it suffices by (1.5) to determine the lineality space of $C(S \cup \{-A_1, \ldots, -A_k\})$. If, however, algorithm (4.4) is used for this purpose, after the columns $-A_1, \ldots, -A_k$ are adjoined to the matrix A, then a matrix results, which is no longer equivalent to A, and each decision must thus start from scratch. This procedure is therefore not suitable for implementation as a subalgorithm.

We proceed to sketch an algorithm which does not require new columns to be adjoined, and which terminates with a matrix that is equivalent to the original one.

A column A_j is *lexico-nonnegative* if $a_{1j} > 0$, or $a_{1j} = 0$ and $a_{2j} > 0$, or $a_{1j} = a_{2j} = 0$ and $a_{3j} > 0$, or . . ., $a_{1j} = \cdots = a_{nj} = 0$. If A_j is lexico-nonnegative and does not vanish, then A_j is *lexico-positive*. We call a matrix *lexico-nonnegative* (*lexico-positive*) if this holds for each column. We then have the following generalization of criterion (4.2):

(5.1) *Suppose* $\hat{\mathrm{A}}$ *arises from* A *by deleting some rows. If* $\hat{\mathrm{A}}$ *is lexico-nonnegative and* $\hat{\mathrm{A}}_j \neq 0$, *then* $\mathrm{A}_j \in \mathrm{S}_C$.

PROOF. Let B be the row that results from combining the rows of A with weights $\epsilon, \epsilon^2, \ldots$. For suitable $\epsilon > 0$ we have $B \geq 0$ with $b_j > 0$ if and only if $\hat{A}_j \neq 0$. Then criterion (4.2) applies to the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$, which is equivalent to A.

We proceed to formulate two criteria on which an algorithm for finding the k-faces of $C(S)$ can be based. Suppose $A_1, \ldots, A_k$ are basic columns. We call the submatrix A of A formed by all those rows in which $A_1, \ldots, A_k$ simultaneously vanish the *complement* of $A_1, \ldots, A_k$. The complement of the entire basis of A is the empty matrix. Schematically, we have

$$A = \left\{ \begin{array}{|c|c|c|} \hline I^{(k)} & 0 & \\ \hline 0 & I^{(m-k)} & \\ \hline \end{array} \right\} = \text{complement}$$

Criterion (5.1) then leads to (see [13]):

(5.2) *If the complement of the basic columns* $A_1, \ldots, A_k$ *is lexico-nonnegative, then* $A_1, \ldots, A_k$ *subdetermine a* k*-face.*

PROOF. It follows from (5.1) that $A_1, \ldots, A_k, -A_1, \ldots, -A_k$ and those elements of S which are linear combinations of the former are the only elements in the lineal part of $S \cup \{-A_1, \ldots, -A_k\}$, and (1.5) applies.—

(5.3) *The basic columns* $A_1, \ldots, A_k$ *do not subdetermine a face if their complement contains a nonpositive nonzero column.*

PROOF. Assume for simplicity that $A_{k+1}, \ldots, A_m$ are the remaining basic columns, and that $a_{ii} = 1$ for $i \leq m$. The complement of $A_1, \ldots, A_k$ then consists of the rows ${}_iA$ with $i > k$. Let now A_c, $c > m$, be a column whose portion in the complement is nonpositive and nonzero. Then

$$A_c = \sum_{i=1}^{k} A_i u_i + \sum_{i=k+1}^{m} A_i w_i,$$

where $w_i = a_{ic} \leq 0$, and, therefore, $A_c \in C((-S) \cup \{A_1, \ldots, A_k\})$. Hence A_c belongs to the lineality space of $C(S \cup \{-A_1, \ldots, -A_k\})$. Now if $A_1, \ldots, A_k$ subdetermine a face, then $L\{A_1, \ldots, A_k\}$ is the lineality space of $C(S \cup \{-A_1, \ldots, -A_k\})$ by (1.5). Thus $A_c \in L\{A_1, \ldots, A_k\}$, which contradicts the fact that not all w_i vanish in the above unique representation of A_c in terms of the basis.—

It is easy to see that algorithm (4.4) and its corresponding primal algorithm can be modified to yield an algorithm whose terminal situations are (5.2) and (5.3). No columns and no nonnegative rows are deleted. A hierarchy of nonegative rows leads eventually to the nonnegative submatrix required by (5.2). The pivot columns vanish in these rows. Hence pivoting does not spoil the nonnegativities already achieved.

6. The 1-Skeleton of $H(\hat{S})$

The problem of finding the vertices of a convex hull $H(\hat{S})$ can be reduced to finding the frame of the cone $C(S)$ where $S := \{A_i = \begin{pmatrix} \hat{A}_i \\ 1 \end{pmatrix} | \hat{A}_i \in \hat{S}\}$. Similarly, the edges of $H(\hat{S})$ correspond to the 2-faces of $C(S)$. There are, however, special features of the problem of finding the vertices and edges of $H(\hat{S})$. First,

(6.1) *If* $\bar{\mathrm{A}}$ *is equivalent to* A, *then* $\bar{\mathrm{A}}_j \in \mathrm{C}(\bar{\mathrm{S}} - \{\mathrm{A}_j\})$ *if and only if* $\bar{\mathrm{A}}_j \in \mathrm{H}(\bar{\mathrm{S}} - \{\bar{\mathrm{A}}_j\})$.

PROOF. If $\bar{A}_j = \sum_{i \neq j} \bar{A}_i u_i$, then by (2.3) $A_j = \sum_{i \neq j} A_i u_i$, which gives $\Sigma u_i = 1$ by considering the last component.

As a consequence of (6.1), a criterion for necessity can be based on the relative magnitudes rather than the signs of the elements of A (Goldstein [7]):

(6.2) *If* $\bar{\mathrm{A}}$ *is equivalent to* A, *and if* $\bar{\mathrm{a}}_{ij}$ *is the unique maximum or minimum of the entries in a row* ${}_i\bar{\mathrm{A}}$ *of* $\bar{\mathrm{A}}$, *then* $\hat{\mathrm{A}}_j$ *is a vertex of* $\mathrm{H}(\hat{\mathrm{S}})$.

Indeed, $\bar{A}_j$ is a vertex of $H(\bar{S})$, and by (6.1) necessary in $\bar{S}$. Criterion (6.2) is valuable in practice since it enables one in general to find some vertices right away as well as during the algorithm. It can be generalized to the case in which the maximum or minimum is not unique. Suppose the maximum of the first row is assumed for the subset $M \subseteq S$. Then a column $A_i \in M$ is necessary in S if it is necessary in M. Hence the examination can be temporarily restricted to M. In particular, criterion (6.2) can be applied to M. For instance, if a_{2s} is the unique maximum or minimum of the entries a_{2j} with $A_j \in M$, then A_s is necessary in S.

Suppose that $\bar{A}$ is in canonical form and equivalent to A. Then the same argument that established (6.2) gives

$$\sum_{i=1}^{m} \bar{a}_{ij} = 1$$

for all j. Thus each column has at least one positive entry. This follows also from the fact that $C(S)$ is pointed i.e., has lineality dimension = 0 (4.2) and that S contains no zero columns.

Suppose S is a frame. Then each nonbasic column has at least one negative and two positive entries by (3.1) and (3.2). If a column has precisely two positive entries, then by (5.3) the corresponding basic columns do not determine an edge.

7. References

[1] Balinsky, M., An algorithm for finding all vertices of convex polyhedral sets, J. Soc. Indust. Appl. Math., **9,** 72–88 (1961).

[2] Charnes, A., Optimality and degeneracy in linear programming, Econometrica **20,** 160–170 (1952).

[3] Dantzig, G. B., Linear Programming and Extensions (Princeton University Press, Princeton, 1963).

[4] Davis, C., Theory of positive linear dependence, Amer. J. Math., **76,** 733–746 (1954).

[5] Fenchel, W., Convex cones, sets and functions, Lecture Notes, Department of Mathematics, Princeton University, Princeton (1953).

[6] Gale, D., On the number of faces of a convex polytope, Can. J. Math. **16,** 12–17 (1964).

[7] Goldstein, A. J., A procedure for determining the convex hull of a set of points of hyperplanes, Comm. ACM, **6,** 356 (1963). Abstract of a paper at 18th Annual ACM Conference 1963.

[8] Gomory, R. E., and H. D. Mills, unpublished manuscript, Princeton (1958).

[9] Klee, V., On the number of vertices of a convex polytope, Can. J. Math. **16,** 701–720 (1964).

[10] Kuhn, H. W., Solvability and consistency for linear equations and inequalities, Amer. Math. Monthly **63,** 217–232 (1956).

[11] Motzkin, T. S., Raiffa, H., Thompson, G., and Thrall, R., The double description method in Contributions to the Theory of Games II, 51–73, Ann. Math. Studies, No. 28 (Princeton University Press, 1953).

[12] Thompson, G. L., Tonge, F. M., and Zionts, S., Techniques for removing nonbinding constraints and extraneous variables from linear programming problems, Management Science **12,** 588–608 (1965).

[13] Wets, R. J.-B. and Witzgall, C., Towards an algebraic characterization of convex polyhedral cones, submitted to a technical Journal.

[14] Farrell, M. J., and M. Fieldhouse, Estimating efficient production functions under increasing returns to scale, J. Roy. Statist. Soc., Ser. A, **125** (II), 252–267 (1962).

(Paper 71B1–189)

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The Coefficients of the Powers of a Polynomial

Morris Newman

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

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It is shown that if $f(z)$ is a polynomial with no zeroes inside the unit circle and if r is any positive number, then the coefficients of $f^r(z)$ tend to zero like n^{-1-r}, and this is best possible.

Key Words: Binomial coefficients, bounds, polynomials.

The general question which prompted this note is the following: Suppose that $f(z)$ is analytic and zero-free inside the unit circle, and normalized so that $f(0)=1$. Let r be real and positive and choose (for definiteness) that determination of $f^r(z)$ for which $f^r(0)=1$. Then what can be said about the order of magnitude of the coefficients of $f^r(z)$? In particular, when will they converge to zero? In such a general setting no simple answer is to be expected and examples illustrating almost any sort of behavior may be given; but the case treated here, namely that when $f(z)$ is a polynomial, is reasonably simple. In fact we shall prove the following theorem:

THEOREM 1. *Let* $f(z)$ *be a polynomial of degree* p *which is zero-free inside the unit circle and such that* $f(0)=1$. *Let* r *be positive and suppose that*

$$f^r(z)=1+\sum_{n=1}^{\infty} a_r(n)z^n$$

Then

$$a_r(n)=0(n^{-1-r}), \qquad n\to\infty, \tag{1}$$

and consequently

$$\lim_{n\to\infty} a_r(n)=0.$$

The constant implied by (1) depends on r and p, of course.

Precise inequalities rather than 0-estimates can be obtained at the expense of some additional complication, but we do not pursue this point here.

The statement is "best possible" in the sense that $a_r(n)$ is of true order of magnitude n^{-1-r} for certain polynomials $f(z)$; for example for $f(z)=1+z$.

The statement is false for negative r. For example if $f(z)=(1-z)^p$ and $r=-2/p$ then $a_r(n)=n+1$.

We require the following lemmas.

LEMMA 1. *Let* r *be arbitrary. Then*

$$\lim_{n\to\infty}(-1)^n\binom{r}{n}n^{1+r}=\frac{1}{\Gamma(-r)}, \tag{2}$$

and so

$$\binom{r}{n}=0(n^{-1-r}), \qquad n\to\infty.$$

Formula (2) is just Euler's definition of the Γ-function. Notice that if r is a positive integer then $\binom{r}{n}$ vanishes for all sufficiently large n, which agrees with the fact that $\Gamma(s)$ has poles at $s=0,-1,-2,\ldots$.

LEMMA 2. *Let* r *be positive. For* $p=1, 2, 3, \ldots$ *define* $t_p(n)$ *by*

$$\sum_{n=0}^{\infty} t_p(n)z^n=\left\{1+\sum_{n=1}^{\infty} n^{-1-r}z^n\right\}^p. \tag{3}$$

Then

$$t_p(n)=0(n^{-1-r}).$$

PROOF. Differentiating (3) and comparing coefficients of corresponding powers of z in the result we obtain the recurrence formula

$$nt_p(n)=p\sum_{k=1}^{n} k^{-r}t_{p-1}(n-k). \tag{4}$$

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The proof is by induction on p. For $p=1$, $t_p(n)=n^{-1-r}$, $n\geq 1$ and so the lemma is certainly true then. Assume the truth of the lemma for $p-1$, $p\geq 2$. Write (4) as

$$nt_p(n)=p\sum_{1\leq k\leq\frac{n-1}{2}}k^{-r}t_{p-1}(n-k)+p\sum_{\frac{n-1}{2}<k\leq n-1}k^{-r}t_{p-1}(n-k)+pn^{-r}.$$

Then

$$\sum_{1\leq k\leq\frac{n-1}{2}}k^{-r}t_{p-1}(n-k)=0\left\{\sum_{1\leq k\leq\frac{n-1}{2}}k^{-r}(n-k)^{-1-r}\right\}$$

$$=0\left\{n^{-1-r}\sum_{1\leq k\leq\frac{n-1}{2}}k^{-r}\right\}=0(n^{-r})$$

(using the crude estimate $k^r\geq 1$), and

$$\sum_{\frac{n-1}{2}<k\leq n-1}k^{-r}t_{p-1}(n-k)=0\left\{\sum_{\frac{n-1}{2}<k\leq n-1}k^{-r}(n-k)^{-1-r}\right\}$$

$$=0\left\{n^{-r}\sum_{\frac{n-1}{2}<k\leq n-1}(n-k)^{-1-r}\right\}=0(n^{-r}),$$

since the series $\sum_{k=1}^{\infty}k^{-1-r}$ converges. Thus

$$nt_p(n)=0(n^{-r}),$$

$$t_p(n)=0(n^{-1-r})$$

and the lemma is proved for p. This concludes the proof of the lemma.

We turn now to the proof of theorem 1. We may write

$$f(z)=(1+\alpha_1 z)(1+\alpha_2 z)\ \ldots\ (1+\alpha_p z),$$

where, because of the assumptions made about $f(z)$, the numbers α_i are all of modulus not exceeding 1. We have

$$f^r(z)=1+\sum_{n=1}^{\infty}a_r(n)z^n$$

$$=\prod_{i=1}^{p}(1+\alpha_i z)^r$$

$$=\prod_{i=1}^{p}\left\{\sum_{n_i=0}^{\infty}\binom{r}{n_i}\alpha_i^{n_i}z^{n_i}\right\}\ll\left\{\sum_{n=0}^{\infty}|\binom{r}{n}|z^n\right\}^p$$

$$\ll K\left\{1+\sum_{n=1}^{\infty}\frac{z^n}{n^{1+r}}\right\}^p$$

$$=K\sum_{n=0}^{\infty}t_p(n)z^n,$$

where K is some suitable positive constant (lemma 1).[1] Hence

$$|a_r(n)|\leq Kt_p(n),$$

and lemma 2 implies the truth of the theorem for all positive r.

It is clear that the critical case is when $f(z)$ has all its roots on the unit circle, since otherwise much stronger inequalities for the coefficients $a_r(n)$ will hold. In fact, we can prove

THEOREM 2. *Let* g(z) *be a polynomial of degree* p *such that* g(0) = 1, *and let* T *be the distance from the origin to the nearest zero of* g(z). *Let* r *be positive,*

$$g^r(z)=1+\sum_{n=1}^{\infty}b_r(n)z^n.$$

Then

$$b_r(n)=0(n^{-1-r}T^{-n}),\qquad n\to\infty.$$

PROOF. It is only necessary to apply theorem 1 to the polynomial $g(Tz)=f(z)$ which now has no zeroes inside the unit circle and satisfies $f(0)=1$.

[1] We write $\sum a_n z^n\ll\sum b_n z^n$ to mean that $b_n\geq 0$ and $|a_n|\leq b_n$.

(Paper 71B1–190)

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Stable Evaluation of Polynomials

C. Mesztenyi* and C. Witzgall**

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A class of Newton forms

$$P(x)=a_0+a_1(x-x_0)+\ \ldots\ +a_n(x-x_0)\ \ldots\ (x-x_{n-1})$$

are discussed which admit a stable evaluation algorithm in an interval $[A, B]$. Stability is defined in the paper. The estimate

$$\frac{\Delta P}{|P|}\leq 2+6\frac{M'(L)L}{M(L)},$$

where $L:=B-A$ and $M(x):=|a_0|+|a_1|x+\ \ldots\ +|a_n|x^n$, is shown to hold for the relative error of evaluation of $P(x)$ in $[A, B]$.

Key Words: Evaluation, Newton form, polynomial, relative error, round-off.

There are algorithms for polynomial evaluation which require fewer operations (Motzkin [4],[1] Belaga [1], Pan [6], Cheney [2], Knuth [3]), but these algorithms are prone to lose significance. The Horner scheme itself leaves much to be desired in this respect, and this fact has led to the study of methods which reduce the loss of significance, but may require additional operations (Rice [7], Lawson [9]).

An evaluation algorithm depends on the form in which the polynomial is given. Forms on which a "stable" evaluation algorithm can be based are called "well-conditioned." This note calls attention to a class of well-conditioned Newton forms.[2] These forms involve more parameters and require more operations for evaluation than the normal form described above. The number of operations is about the same as for evaluating an expansion by Tchebychef polynomials, but in general more parameters are required.

Loosely speaking, an algorithm is "unstable" if one observes for some arguments an excessive loss of significance. A rigorous, but somewhat narrow, definition of stability will be based on an idealized floating point arithmetic and an error analysis similar to the one carried out by v. Neumann and Goldstine [5], however referring to the relative rather than the absolute error. The arithmetic will be finite in that it will restrict itself to numbers which are "representable" with mantissae of a fixed finite length.[3] It will be infinite in that it will permit arbitrarily large positive and negative exponents. Thus there will be infinitely many representable numbers. Some of the theoretical results of this paper hinge on this idealization, and might be considered artificial. Nevertheless, these results point in the direction of desirable improvements, and provide a reasonable classification of algorithms from the view point of stability.

Our definition of condition and stability differs from that given by Rice [7].

1. Minimal Newton forms. The evaluation algorithms to be considered are based on the

Newton form

of polynomials:

$$P(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+\ \ldots\ +a_n(x-x_0)\ \ldots\ (x-x_{n-1}). \qquad (1.1)$$

Polynomials in this form are evaluated by the following adaption of the Horner scheme:

$$\begin{aligned} &D_n:=a_n\\ &D_i:=a_i+(x-x_i)D_{i+1}, \qquad i=n-1,\ \ldots\ ,0,\\ &P(x):=D_0. \end{aligned} \qquad (1.2)$$

*University of Maryland, College Park, Md.
**Mathematics Research Laboratory, Boeing Scientific Laboratories, Seattle, Washington 98110.

[1] Figures in brackets indicate the literature references at the end of this paper.
[2] Our study of these forms was stimulated by a discussion with J. Rice on polynomial evaluation. It profited from discussions with A. J. Goldman, who also detected a major mistake in the previous version of this paper.

[3] $0.53_{10}15$ is representable in decimal arithmetic of mantissa length $s\geq 2$; 0.1_{10} is not representable in any binary arithmetic.

Instability can arise when adding the coefficients a_i and when subtracting the "critical values" x_i. There are three reasons for being less concerned with the effect of the subtractions:

First, if the values x and x_i are both representable in the given finite arithmetic, then forming the differences $x-x_i$ is a "stable" operation. A similar advantage does not obtain for the addition, even if the coefficients a_i are representable. Furthermore, the effect of replacing x and x_i by representable numbers close to them, in other words, the effect of "rounding," can be estimated by means of the differential formula:

$$dP=[D_1]d(x-x_0)+[D_2(x-x_0)]d(x-x_1)+\ldots$$

$$\ldots+[D_n(x-x_0)\ldots(x-x_{n-2})]d(x-x_n).$$

Here the D_i are the intermediate results of (1.2).

Second, if the terms of the sum $D_i:=a_i+(x-x_i)D_{i+1}$ are of the same sign, then the effect of the relative error of $(x-x_i)D_{i+1}$ on the relative error of D_i will be weighed by the ratio $|(x-x_i)D_{i+1}/D_i|$. The relative error of $x-x_i$ tends to be large when $|x-x_i|$ becomes small. Thus the relative error caused by forming the difference is toned down by the above ratio precisely when it tends to be high. On the other hand, if x and $-x_i$ have the same sign, then one does not observe a similar toning down of the relative error incurred by the preceding addition.

Third, we give a theoretical reason. We shall be able to show that $x-x_i$ can be evaluated "stably" if additional coefficients and operations are introduced, provided x is representable, even if x_i is not representable.

Recognizing the additions as the main cause of instability, the idea is to neutralize them by choosing the critical values x_i so as to ensure that a_i has always the same sign as $(x-x_i)D_{i+1}$.

This is indeed possible in any given interval of evaluation $[A, B]$. The first step of constructing such a Newton form is to divide out all zeros of the polynomial $P(x)$ in $[A, B]$:

$$P(x)=:(x-x_0)\ \ldots\ (x-x_m)P^*(x).$$

One then defines $a_0=a_1=\ldots=a_m=0$, and the problem is clearly reduced to finding a suitable Newton form for $P^*(x)$.

Let us therefore assume that $P(x)\neq 0$ in $[A, B]$. If $P(x)$ is given in Newton form (1.1), and if a_0 has the same sign as the sum of the remaining terms for all x in $[A, B]$, then sign $(a_0)=$ sign $(P(x))$, and $|a_0|\leq|P(x)|$ in $[A, B]$. This suggests the following definition of a_0:

$$|a_0|:=\min\ \{|P(x)|\mid x\epsilon[A, B]\}$$

$$\text{sign}\,(a_0):=\text{sign}\,(P(x)).$$

Then $P(x)-a_0$ has zeroes

$$x_0, x_1, \ldots, x_{k-1}$$

in $[A, B]$. Zeroes are represented as often as their multiplicity indicates. If, for instance, y is a double zero of $P(x)$ in $[A, B]$ then y occurs twice among the x_i. We have

$$P(x)=D_0(x)=a_0+(x-x_0)\ldots(x-x_{k-1})D_k(x).$$

The polynomial $D_k(x)$ does not vanish in $[A, B]$, and can therefore be treated in the same way as $P(x)$. The coefficient a_k is defined by

$$|a_k|:=\min\ \{|D_k(x)|\mid x\epsilon[A, B]\}$$
$$\text{sign}\,(a_k):=\text{sign}\,(D_k(x)),$$

and

$$x_k, \ldots, x_{l-1}$$

are the zeroes of $D_k(x)-a_k$. We have again

$$D_k(x)=a_k+(x-x_k)\ \ldots\ (x-x_{l-1})D(x),$$

and so on. We call the Newton form so constructed the

minimal Newton form

of the polynomial $P(x)$ in $[A, B]$.

If all zeros of $P(x)$ are in $[A, B]$, then the minimal Newton form of $P(x)$ in $[A, B]$ coincides with the "product form" $a_n\Pi(x-x_i)$ of the polynomial. If $P(x)$ does not vanish in $[A, B]$, then all zeros of odd multiplicity encountered in the construction of the minimal Newton form are at the ends of the interval $[A, B]$.

We shall show later that a minimal Newton form can be evaluated in a "stable" manner if the argument x is representable.

2. On transformations into stable forms. It is quite clear that the process of determining the minimal Newton form is, in itself, a very unstable process, which has to be carried out in double or even triple precision. This appears to be a principle: the constants of a better conditioned form contain more "information," and if one wants this additional information, then one has to work for it.

For this reason, we shall not concern ourselves with the difficulties of finding the constants of algorithms. These difficulties are bound to increase with the quality of the end-product. Besides, minimal Newton forms will be used only if a polynomial has to be evaluated repeatedly, as for instance in function subroutines.

3. Evaluation of derivatives. The Horner scheme is frequently extended to determine the value of the first derivative of a polynomial, using the algorithm:

$$E_n:=D_n$$

$$E_i:=D_i+xD_{i+1},\qquad i=n-1, \ldots, 1, \qquad (3.1)$$

$P'(x) := E_1$,

where the D_i denote the intermediate results of the Horner scheme for $P(x)$. We shall use this fact in section 11 for deriving an error estimate.

Similar algorithms exist for Newton forms. If the D_i are the intermediate results of algorithm (1.2) applied for a given x, then:

$$P'(x) = D_1 + D_2(x - x_0) + \ldots + D_n(x - x_0) \ldots (x - x_{n-2}),$$

which suggests the algorithm:

$$E_n := D_n$$

$$E_i := D_i + (x - x_{i-1})E_{i+1}, \qquad i = n-1, \ldots, 1$$

$$P'(x) := E_1.$$

This algorithm is in general not stable, even if the original polynomial P is in minimal Newton form.

4. Numerical example.[4] Consider the ill-conditioned polynomial $a_0 + \ldots + x^5$ with the coefficients:

$$\begin{aligned} a_0 &= 4.\ 10074\ 70239\ 8387 \\ a_1 &= -11.\ 29173\ 84073\ 737 \\ a_2 &= 8.\ 42475\ 03796\ 1924 \\ a_3 &= 0.\ 92113\ 31318\ 58071 \\ a_4 &= -3.\ 05937\ 81605\ 8204. \end{aligned}$$

Its minimal Newton form in the interval [0, 1] has the following coefficients and critical values:

$$\begin{aligned} b_0 &= 0.00103\ 19917\ 44066\ 05 \\ b_1 &= 0.0 \\ b_2 &= 3.41269\ 84126\ 9841 \\ b_3 &= -1.87912\ 08791\ 2088 \\ b_4 &= 0.60784\ 31372\ 54902 \\ b_5 &= 1.0 \\ x_0 &= 0.83361\ 06489\ 18469 \\ x_1 &= x_0 \\ x_2 &= x_3 = 1.0 \\ x_4 &= 0.0. \end{aligned}$$

To compare the loss of significance incurred in evaluating the polynomial in normal form and minimal Newton form, the interval [0, 1] was divided into 50 subintervals, in each of which 50 random arguments were generated. The maximum number of significant digits lost was plotted for each subinterval (fig. 1).

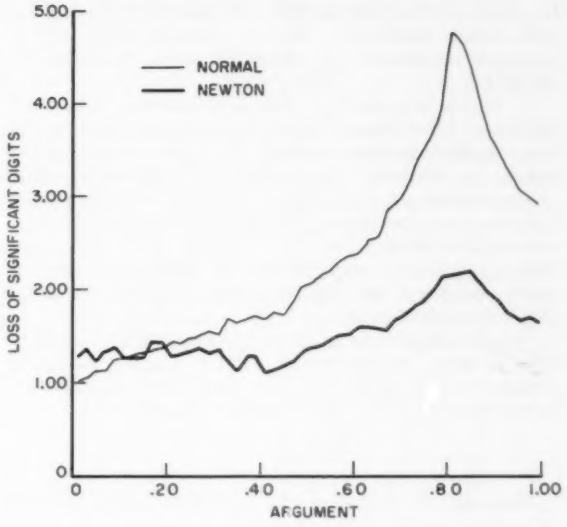

FIGURE 1

5. Representable numbers. This and the following sections will contain some theoretical examinations. An arithmetic will be specified, with respect to which we shall define stability. Actual error estimates are then derived for the evaluation of minimal Newton forms.

Consider a floating point arithmetic with number base β and mantissa length s, and assume that there are no restrictions on the size of the exponents. Denote by

$$\Sigma$$

the set of all numbers which are representable in this arithmetic. To each real number x, we assign a number $\bar{x} \epsilon \Sigma$ such that no other number in Σ is closer to x. Thus

$$|x - \bar{x}| \leq |x - y| \text{ for all } y \epsilon \Sigma. \tag{5.1}$$

In particular, $\bar{x} = x$ if $x \epsilon \Sigma$. We assume that $\overline{(-x)} = -\bar{x}$, therefore $-\Sigma = \Sigma$.

With the notation

$$\alpha(x) := \begin{cases} 0 \text{ if } x \epsilon \Sigma \\ 1 \text{ otherwise} \end{cases}$$

one verifies that[5]

$$|x - \bar{x}| \leq |\bar{x}|\alpha(x)\epsilon \leq |\bar{x}|\epsilon \text{ where } \epsilon := \frac{1}{2}\beta^{-s+1} \tag{5.2}$$

(Scarborough's Theorem I [8] p. 3). Note that (5.2) holds only if the arithmetic does not restrict the size of the exponents. Indeed, it follows from (5.2) that:

$$\bar{x} = 0 \text{ implies } x = 0. \tag{5.3}$$

[4] The computer time for this calculation was supported by the Computer Science Center of the University of Maryland under Grant NsG 398 from the National Aeronautics and Space Administration.

[5] Actually a sharper estimate holds: $|x - \bar{x}| \leq \epsilon\beta^t$ where $t := \lfloor \log_\beta |\bar{x}| \rfloor$, that is, t is the greatest integer smaller than $\log_\beta |\bar{x}|$.

In other words, zero is the only number which gives zero when rounded. This is clearly only true if negative exponents of arbitrarily large modulus are admitted.

6. Stable algorithms. We now assume that each addition, subtraction, multiplication, and division of two numbers in Σ is carried out exactly, and that the result is rounded afterwards. Furthermore, each given number which is not in Σ must be rounded before it enters an operation. These rules appear to be a reasonable idealization of a floating point arithmetic which combines two registers to hold the result of each operation, normalizing and rounding to single precision subsequently.

An algorithm aims at computing a result r as a function of given parameters $a, b, c, \ldots$, but in reality computes a quantity $\hat{r}$. The error that is generated in the course of the algorithm then is defined by

$$\Delta r := |r - \hat{r}|.$$

Consider, for instance, the algorithm which consists of multiplying two given numbers u and v:

$$p := uv$$

Then $\hat{p} := \overline{\overline{u}\,\overline{v}}$, and (5.2) gives

$$|p - \hat{p}| = |uv - u\bar{v} + u\bar{v} - \bar{u}\bar{v} + \bar{u}\bar{v} - \hat{p}|$$

$$\leq |u\bar{v}|\epsilon + |\bar{u}\bar{v}|\epsilon + |\hat{p}|\epsilon.$$

In view of

$$|u| \leq |\bar{u}| + |u - \bar{u}| \leq (1+\epsilon)|\bar{u}|,$$

$$|\bar{u}\bar{v}| \leq |\hat{p}| + |\bar{u}\bar{v} - \hat{p}| \leq (1+\epsilon)|\hat{p}|,$$

one has

$$\frac{\Delta p}{\epsilon|\hat{p}|} = \frac{|p - \hat{p}|}{\epsilon|\hat{p}|} \leq 3 + 3\epsilon + \epsilon^2. \tag{6.1}$$

The upper bound of the above error term does not depend on u and v. Hence we say that the formation of the product uv from given numbers u and v is "stable."

In general, we call an algorithm which computes r from given parameters $a, b, c, \ldots$

stable,

if there exists an error estimate of the form

$$\frac{\Delta r}{\epsilon|\hat{r}|} \leq k,$$

where k does *not depend* on the values of the parameters $a, b, c, \ldots$

We turn now to the addition of two given numbers u and v:

$$s = u + v.$$

Then $\hat{s} = \overline{\bar{u} + \bar{v}}$, and one has after a similar calculation:

$$\frac{\Delta s}{\epsilon|\hat{s}|} \leq 1 + \frac{|\bar{u}|}{|\hat{s}|}\alpha(u) + \frac{|\bar{v}|}{|\hat{s}|}\alpha(v). \tag{6.2}$$

The estimate (6.2) can be improved. For instance, the first term in (6.2) can be deleted if $|s| \leq \min\{|u|, |v|\}$. However, no matter how refined the estimate, it will always depend on u and v. Indeed, $u + v \neq 0$ and $\hat{s} = 0$ may both hold simultaneously, precluding any finite bound. The addition of two numbers is therefore not stable.

In the special case $\alpha(u) = \alpha(v) = 0$, that is, if both u and v are representable, then (6.2) proves the addition to be stable. This is also true if u and v have the same sign.

7. Simplifying error estimates. Henceforth we shall assume that ϵ is very small. Thus we shall delete ϵ in expressions of the form $k + l\epsilon$, if both k and l are *parameter independent:*

$$k + l\epsilon \cong k. \tag{7.1}$$

This leads us, for instance, to equate $|\hat{r}|$ with $|r|$, since

$$|r| \geq |\hat{r}| - |r - \hat{r}| \geq |\hat{r}| - k\epsilon|\hat{r}| = |\hat{r}|(1 - k\epsilon)$$

and similarly $|\hat{r}| \geq |r|(1 - k\epsilon)$, provided the algorithm producing $\hat{r}$ is stable.

We state without proof that, if an algorithm is shown stable with the help of rule (7.1), then it could have been shown stable without it.

8. Error propagation. Suppose that p, u, and v are intermediate results of some algorithm, and that

$$p = uv$$

according to this algorithm. We want to estimate the error of p in terms of Δu and Δv, and the error generated by rounding and multiplying. Now $\hat{p} = \hat{\hat{u}}\hat{\hat{v}}$, and analogous to the derivation of (6.1) we have

$$\Delta p \leq |v|\Delta u + |\hat{u}|\Delta v + |\hat{p}|\epsilon.$$

If the intermediate results u and v were arrived at in a stable manner, then this is also true for p. By (7.1) we can therefore consider $|u|$ and $|\hat{u}|$, as well as $|p|$ and $|\hat{p}|$, to be equal. Similar considerations hold for the division of two intermediate results. We conclude: If u and v are intermediate results of the algorithm which are obtained in a stable manner, then

$$\frac{\Delta uv^{\pm 1}}{\epsilon|uv^{\pm 1}|} \leq 1 + \frac{\Delta u}{\epsilon|u|} + \frac{\Delta v}{\epsilon|v|}. \tag{8.1}$$

The estimate (8.1) is usually postulated without assumptions about the algorithm. It is argued that $\frac{\Delta u}{|u|}$ and $\frac{\Delta v}{|v|}$ are comparable to ϵ in magnitude, . . . a claim that comes close to assuming stability of the algorithm . . . , and that therefore the term $\frac{\Delta u}{\epsilon|u|}\cdot\frac{\Delta v}{\epsilon|v|}\epsilon$ may be neglected.

The following estimate can be derived without stability assumptions:

$$\frac{\Delta(u\pm v)}{\epsilon|u\pm v|}\leqslant 1+\frac{|u|}{|u\pm v|}\cdot\frac{\Delta u}{\epsilon|u|}+\frac{|v|}{|u\pm v|}\cdot\frac{\Delta v}{\epsilon|v|}. \tag{8.2}$$

9. Stability of minimal Newton forms. Let us now turn to algorithms for the evaluation of polynomials. Once and for all we will assume that

(9.1) *the argument* x *for which a polynomial* P(x) *is to be evaluated is representable.*

Consider the evaluation of a polynomial in minimal Newton form by the adapted Horner scheme (1.2). All operations in this algorithm are stable except the subtractions $x-x_i$. The existence of a stable evaluation thus depends on the possibility of evaluating the differences $x-x_i$ in a stable manner.

To evaluate the difference $x-y$, we split y into its representable part and remainder

$$y=\bar{y}+r,$$

and evaluate $(x-\bar{y})-r$. Since x and $\bar{y}$ are both representable, and therefore $\Delta x=\Delta\bar{y}=0$, we have by (8.2):

$$\frac{\Delta(x-\bar{y})}{\epsilon|x-y|}\leqslant 1.$$

Then again by (8.2):

$$\frac{\Delta(x-y)}{\epsilon|x-y|}\leqslant 1+\frac{|x-\bar{y}|}{|x-y|}\cdot\frac{\Delta(x-\bar{y})}{\epsilon|x-\bar{y}|}+\frac{|r|}{|x-y|}\cdot\frac{\Delta r}{\epsilon|r|}\leqslant 1+\frac{|x-\bar{y}|+|r|}{|x-y|}.$$

In view of $|x-\bar{y}|=|x-y+r|\leqslant|x-y|+|r|$ this gives

$$\frac{\Delta(x-y)}{\epsilon|x-y|}\leqslant 2+\frac{2|r|}{|x-y|}.$$

Now according to (5.1)

$$|r|=|\bar{y}-y|\leqslant|x-y|,$$

since x is representable. Hence

$$\frac{\Delta(x-y)}{\epsilon|x-y|}\leqslant 4. \tag{9.2}$$

The algorithms $((\bar{x}-\bar{y})+s)-r$ and $(\bar{x}-\bar{y})+(s-r)$, where x is not required to be representable and where s denotes the remainder term $x-\bar{x}$ of x, are not stable. Indeed, if $\bar{x}=\bar{y}$, then both algorithms consist of simply subtracting r from s, which was seen to be unstable in section 6.

Consider the algorithm (1.2) for evaluating Newton forms. Since all other operations of (1.2) are stable, the entire algorithm (1.2) is stable provided the differences $x-x_i$ are evaluated in the stable manner described above.

For the step

$$D_i:=a_i+(x-x_i)D_{i+1},$$

we have by (9.2), (8.1), and (8.2)

$$\frac{\Delta(x-x_i)D_{i+1}}{\epsilon|(x-x_i)D_{i+1}|}\leqslant 1+\frac{\Delta(x-x_i)}{\epsilon|x-x_i|}+\frac{\Delta D_{i+1}}{\epsilon|D_{i+1}|}\leqslant 5+\frac{\Delta D_{i+1}}{\epsilon|D_{i+1}|}.$$

$$\frac{\Delta D_i}{\epsilon|D_i|}\leqslant 1+\frac{|a_i|}{|D_i|}\cdot\frac{\Delta a_i}{\epsilon|a_i|}+\frac{|(x-x_i)D_{i+1}|}{|D_i|}\cdot\frac{\Delta(x-x_i)D_{i+1}}{\epsilon|(x-x_i)D_{i+1}|}$$

$$\leqslant 1+\frac{|a_i|}{|D_i|}+\frac{|(x-x_i)D_{i+1}|}{|D_i|}\left(5+\frac{\Delta D_{i+1}}{\epsilon|D_{i+1}|}\right)$$

$$\leqslant 1+\frac{|a_i|+|(x-x_i)D_{i+1}|}{|D_i|}+\frac{|(x-x_i)D_{i+1}|}{|D_i|}\left(4+\frac{\Delta D_{i+1}}{\epsilon|D_{i+1}|}\right).$$

Since minimal Newton forms are constructed so as to ensure that $|a_i|+|(x-x_i)D_{i+1}|=|D_i|$, we finally have

$$\frac{\Delta D_i}{\epsilon|D_i|}\leqslant 2+\frac{|(x-x_i)\Delta_{i+1}|}{|D_i|}\left(4+\frac{\Delta D_{i+1}}{\epsilon|D_{i+1}|}\right), \tag{9.3}$$

from which one immediately concludes that

$$\frac{\Delta D_i}{\epsilon|D_i|}\leqslant 6+\frac{\Delta D_{i+1}}{\epsilon|D_{i+1}|}.$$

Since $\Delta D_n\leqslant|D_n|\epsilon$, we have finally

$$\frac{\Delta P}{\epsilon|P|}\leqslant 6n+1, \tag{9.4}$$

where n is the degree of the polynomial $P(x)$.

Note that application of the estimate (8.2) is justified since the factors of the products $(x-x_i)D_{i+1}$ have been arrived at in a stable manner.

10. Product forms. Every polynomial with real coefficients can be written in product form:

$$P(x)=a_n\prod_{i=0}^{k-1}(x-x_i)\prod_{i=k}^{k+l-1}(d_i+(x-x_i)^2),$$

$$d_i>0, \qquad k+2l=n. \qquad (10.1)$$

A stable evaluation algorithm results if the differences $x-x_i$ are evaluated in the stable manner described in section 9. The algorithm involves not more than $2n+1$ parameters, $2n$ additions, and n multiplications. The following error estimate follows from (9.2), (8.1), and (8.2):

$$\frac{\Delta P}{\epsilon|P|}\leq 5k+11l+1\leq 6n. \qquad (10.2)$$

11. Improved estimate. The reader observes not only, that the error estimate (10.2) is better than the estimate (9.4) for minimal Newton forms, but also that fewer operations are required. It is therefore questionable whether minimal Newton forms should be considered at all, unless the error estimate (9.4) can be improved. We proceed to show that this is indeed possible.

Let L denote the length of the interval $[A, B]$ and define

$$B_i:=|a_i|+|a_{i+1}|L+\ldots+|a_n|L^{n-i}, \qquad i=0, \ldots, n.$$

Then $|D_i|\leq B_i$ in $[A, B]$ since

$$D_i=a_i+a_{i+1}(x-x_i)+\ldots+a_n(x-x_i)\ldots(x-x_{n-1}).$$

Therefore

$$\frac{|(x-x_i)D_{i+1}|}{|D_i|}=1-\frac{|a_i|}{|D_i|}\leq 1-\frac{|a_i|}{B_i}=\frac{B_{i+1}}{B_i}L$$

and (9.3) reduces to

$$\frac{\Delta D_i}{\epsilon|D_i|}\leq 2+\frac{B_{i+1}}{B_i}\left(4+\frac{\Delta D_{i+1}}{\epsilon|D_{i+1}|}\right)L.$$

Consequently

$$\frac{\Delta P}{\epsilon|P|}\leq 2+\frac{6B_1}{B_0}L+\frac{6B_1B_2}{B_0B_1}L^2+\ldots+\frac{B_1\ldots B_n}{B_0\ldots B_{n-1}}L^n,$$

and therefore

$$\frac{\Delta P}{\epsilon|P|}\leq 2+\frac{6}{B_0}(B_1L+B_2L^2+\ldots+B_nL^n). \qquad (11.1)$$

If we introduce the polynomial

$$M(L):=|a_0|+|a_1|L+|a_2|L^2+\ldots+|a_n|L^n,$$

Then the B_i are the intermediate results of the Horner scheme applied to $M(L)$. Thus $B_0=M(L)$. Moreover, we obtain from (3.1) that $B_1+\ldots+B_nL^{n-1}=M'(L)$, and we may rewrite (11.1) as follows:

$$\frac{\Delta P}{\epsilon|P|}\leq 2+6\frac{M'(L)L}{M(L)}. \qquad (11.2)$$

This error estimate establishes a preference for minimal Newton forms in small intervals.

Polynomials with nonnegative coefficients are minimal Newton forms for all intervals $[0, x]$ with $x>0$. For such polynomials one derives analogously

$$\frac{\Delta P}{\epsilon|P|}\leq 2+2\frac{P'(L)L}{P(L)}.$$

12. Approximately minimal Newton forms. Each sequence of n critical values x_i determines a unique Newton representation of a given polynomial $P(x)$. If the values x_i^* approximate the points x_i which determine the minimal Newton form, then the Newton form

$$P(x)=a_0^*+a_1^*(x-x_0^*)+\ldots$$

$$+a_n^*(x-x_0^*)\ldots(x-x_{n-1}^*),$$

which is determined by the values x_i^* is approximately minimal, and for all practical purposes, it is stable.

For certain polynomials of low degree approximately minimal Newton forms may be even rigorously stable in the sense of section 6. Consider for instance the quadratic polynomial in minimal Newton form

$$Q(x)=a_0+(x-z)^2, \qquad a_0>0.$$

The Newton representation of $Q(x)$ determined by the rounded value $\bar{z}$ takes the form:

$$Q(x)=b_0+b_1(x-\bar{z})+(x-\bar{z})^2, \qquad (12.1)$$

where $b_0>r^2$, $b_1=-2r$, with $r:=z-\bar{z}$. We proceed to prove that the evaluation of (12.1) by algorithm (1.2) is stable. We put $s:=x-\bar{z}$. Then

$$D:=b_1+(x-\bar{z})=-2r+s$$
$$Q=b_0+(x-\bar{z})D=b_0+sD.$$

We have

$$\Delta D\leq 2|r|\epsilon+s|\epsilon|+|D|\epsilon$$

$$\Delta sD\leq 2|sD|\epsilon+|s|\Delta D\leq 3|sD|\epsilon+2|rs|\epsilon+s^2\epsilon$$

and

$$\Delta Q \leq |b_0|\epsilon + 3|sD|\epsilon + 2|rs|\epsilon + s^2\epsilon + Q\epsilon \qquad (12.2)$$

By (5.1), $|s-r| = |x-z| \geq |\bar{z}-z| = |r|$. Hence

$$Q = a_0 + (x-z)^2 = (b_0 - r^2) + (s-r)^2 \geq b_0$$

for all representable x. This in turn implies $Q \geq sD \geq 0$. As a consequence, we may delete most of the absolute bars in (12.2):

$$\frac{\Delta Q}{\epsilon} \leq b_0 + 3sD + 2|rs| + s^2 + Q \leq 2Q + 2sD + 2|rs| + s^2.$$

If $rs \leq 0$, then $2|rs| + s^2 = s^2 - 2rs = sD$. Hence

$$\frac{\Delta Q}{\epsilon} \leq 2Q + 3sD \leq 5Q.$$

If $rs > 0$, then $|s| \geq 2|r|$. Indeed $(s-r)^2 \geq r^2$ implies $2rs \leq s^2$, and if $rs > 0$, then we have also $2|r||s| \leq |s|^2$. Furthermore, $sD + 2|rs| = sD + 2rs = s^2$. Finally, we note that

$$\frac{s^2}{Q} \leq \frac{s^2}{(s-r)^2} \leq 4 \text{ for } |s| \geq |2r|.$$

Hence

$$\frac{\Delta Q}{\epsilon |Q|} \leq 2 + \frac{sD}{Q} + \frac{2s^2}{(s-r)^2} \leq 11.$$

This estimate is as good as the one derived from (10.2) taking into account the fact that $a_2 = 1$.

References

[1] Belaga, E. G., Some problems involved in the computation of polynomials, Dokl. Adad. Nauk SSR **123**, 775–777 (1958).

[2] Cheney, E. W., Algorithms for the evaluation of polynomials using a minimum number of multiplications, Tech. Note 2, Computation and Data Processing Center, Aerospace Corp., 1962.

[3] Knuth, D. E., Evaluation of polynomials by computer, Comm. Assoc. Comp. Mach. **5**, 595–599 (1962).

[4] Motzkin, T. S., Evaluation of polynomials, Bull. Amer. Math. Soc. **61**, 163, Abstract 315B (1955).

[5] v. Neumann, J., and Goldstine, Numerical inverting of matrices of high order, Bull. Amer. Math. Soc. **53**, 1021–1099 (1947).

[6] Pan, V. Ya., Certain schemes for the calculation of values of polynomials with real coefficients, Prob. Kibernetiki **5**, 17–29 (1959).

[7] Rice, J. R., On the Conditioning of Polynomial and Rational Forms.

[8] Scarbrough, J. B., Numerical Mathematical Analysis (Johns Hopkins Press, Baltimore, Md., 1930).

[9] Lawson, C. L., Study of methods of polynomial evaluations, Jet Propulsion Laboratory, Interim Report No. 1, Feb. 1964.

FIGURE 1

(Paper 71B1–191)

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On Involutions

O. Shisha* and C. B. Mehr**

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Two methods are described of constructing real functions over the reals which are one-to-one, assume every real value and are their own inverses, and several examples are given. It is also shown that such a function, if everywhere continuous, is either the function $f(x) \equiv x$ or else is strictly decreasing.

Key Words: Inverses, involutions, real functions.

1. We shall consider real functions f whose domain is the set of real numbers, which take on every real value, are one-to-one, and satisfy for every real x, $f^{-1}(x) = f(x)$, where f^{-1} is the inverse function of f. We denote by I the set of all such functions. Recall that functions which are their own inverses are called *involutions*.

Suppose that a real function f has as its domain the set of real numbers. Then it belongs to I if and only if

$$f(f(x)) = x \text{ for every real } x. \tag{1}$$

Indeed, if $f \in I$, then for every real x, $f(f(x)) = f(f^{-1}(x)) = x$. Conversely, if (1) holds, then f takes on every real value, is one-to-one (for $f(x_1) = f(x_2)$ implies $x_1 = f(f(x_1)) = f(f(x_2)) = x_2$), and for every real x, $f^{-1}(x) = f(x)$.

Note that the graph of every f in I is symmetric in the line $y = x$.

Conversely, if G is a set in the x, y plane, symmetric in the line $y = x$ and containing, for every real x, a unique point whose abscissa is x, then G is the graph of a function belonging to I.

2. One way of obtaining functions in I is the following. Start with a real function $g(x, y)$ whose domain is the set of all ordered pairs of real numbers, and which is such that $g(x, y) = 0$ implies $g(y, x) = 0$. (This property holds, e.g., if g is symmetric, i.e., if for every real x, y, we have $g(y, x) = g(x, y)$.) Suppose that for every real x, there is a unique real y (to be denoted $f(x)$) such that $g(x, y) = 0$. Then f (with domain the set of reals) belongs to I. Indeed, for every real x,

$$g(f(x), x) = g(x, f(x)) = 0,$$

and consequently $f(f(x)) = x$.

EXAMPLE 1. Let $g(x, y) \equiv x + y - c$, c being an arbitrary real constant. We obtain from it the function $f(x) \equiv c - x$ belonging to I.

EXAMPLE 2. Let $g(x, y) \equiv x - y$. The corresponding $f \in I$ is $f(x) \equiv x$.

EXAMPLE 3. Let $g(x, y) \equiv x^3 + y^3 - c$, c being an arbitrary real constant. We get from it the function $f(x) \equiv \sqrt[3]{c - x^3} \in I$.

3. Another method of obtaining functions in I is based on the last paragraph of section 1. We illustrate this method by the following

EXAMPLE 4. In the X, Y plane consider the hyperbola $X^2 - \frac{1}{2}Y^2 = 1$. Let R, L denote, respectively, its right-hand and left-hand branches. Consider now a new coordinate system, x, y, obtained from the X, Y system by a clockwise rotation of 45°. In the new coordinate system the equation of R (which is symmetric in the line $y = x$) is

$$y = -3x + 2(2x^2 + 1)^{1/2}.$$

Thus, $f(x) \equiv -3x + 2(2x^2 + 1)^{1/2}$ belongs to I. Similarly, the equation of L in the new coordinate system is

$$y = -3x - 2(2x^2 + 1)^{1/2},$$

and consequently, $f(x) \equiv -3x - 2(2x^2 + 1)^{1/2}$ belongs to I.

4. Consider the functions in I which are everywhere continuous. Since such a function takes on every real value exactly once, it must be, throughout the real line, either strictly increasing or strictly decreasing. For example, $f(x) \equiv x$ is a function in I which is strictly increasing. It is interesting to note that *all other everywhere continuous functions in I are strictly decreasing.* Indeed, let $F(x) (\not\equiv x)$ be an everywhere continuous function in I. Then its graph contains two points which do not lie on the line $y = x$, but which are symmetric in this line. Let (x_1, y_1), (x_2, y_2) (with $x_1 < x_2$) be such points. Then $y_1 > y_2$ (draw a figure!). So $F(x_1) > F(x_2)$, and consequently, F is strictly decreasing.

*Aerospace Research Laboratories, Wright-Patterson AFB, Ohio 45433.
**The contribution of this author was made while he was working at Aerospace Research Laboratories, Wright-Patterson AFB, Ohio. Present address: Ohio University, Athens, Ohio 45701.

5. Let us examine the smoothness of the various examples we have of functions belonging to I. The functions $c-x$ and x of Examples 1 and 2 are differentiable throughout the real line; in fact they are analytic at each real point. The function $f(x)\equiv\sqrt[3]{c-x^3}$ of Example 3 is everywhere continuous. If $c=0$, it reduces to $-x$. Otherwise, it is everywhere differentiable except at the point $x=\sqrt[3]{c}$, where it is not.

Let us now look at the functions f of Example 4. The function $2z^2+1$ of the complex variable z vanishes at $2^{-1/2}i$, $-2^{-1/2}i$ and nowhere else. Consequently, the real functions $-3x+2(2x^2+1)^{1/2}$, $-3x-2(2x^2+1)^{1/2}$ of Example 4, are analytic at every point of the x axis.

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Minimum Number of Subsets to Distinguish Individual Elements

P. R. Meyers

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

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Given a set S of cardinality m, we determine the minimum cardinality $f(m)$ for a family F of subsets of S such that each $s \epsilon S$ can be expressed as the intersection of some subfamily of F. The problem is solved in the following inverse form. For a given number n of subsets of S, find $g(n)$: the maximum number of elements of S which can be written as the intersection of some of these subsets. We show that $g(n)$ is the largest binomial coefficient for combinations of n things.

Key Words: Classification design, combinatorics, set theory.

1. Introduction

Let S be a finite set of given cardinality $|S| = m$. An element $s \epsilon S$ will be said to be *distinguished* by a family $\mathscr{F}$ of subsets of S, if $\{s\}$ is the intersection of some subfamily of $\mathscr{F}$. In this note we solve the following combinatorial problem (conveyed by K. E. Kloss): What is the minimum possible cardinality $f(m)$ for a family which distinguishes *all* elements of S? (Trivially $f(m) \leq m$, since

$$\mathscr{F} = \{\{s\} : s \epsilon S\}$$

distinguishes all elements.)

The question may sound like another one which arises fairly naturally in a context of classification design or information retrieval: How many "categories" (subsets) must be established so that any item (element) in a collection can be uniquely specified by listing those categories under which it falls? The categories which uniquely specify some item may be a subcollection of those which specify another item, while any family of subsets with a one element intersection cannot be part of a larger family with a different nonempty intersection.

It will be more convenient to work with the following *inverse* form of the problem: to determine $g(n)$, the maximum cardinality of a set S of elements which are distinguished by some family $\mathscr{F} = \{F_1, \ldots, F_n\}$. The inversion is made precise by removing from each F_i those elements of S which are not distinguished by $\mathscr{F}$. (For this question n is fixed, but not $\mathscr{F}$.) It will be shown below that

$$g(n) = \binom{n}{[n/2]}, \tag{1}$$

where $(\cdot)$ is the binomial coefficient and $[n/2]$ is the largest integer not greater than $n/2$. This yields an implicit solution to the original problem, since

$$f(m) = \min \{n : g(n) \geq m\}, \tag{2}$$

follows when we observe that

$$m \leq g(n) \text{ iff } f(m) \leq n. \tag{3}$$

2. Proof

Let $h(n)$ be the binomial coefficient on the right-hand side of (1). We first show that $h(n) \leq g(n)$. For this purpose, let A be a set with cardinality $|A| = n$ and let

$$S^* = \{s_1, \ldots, s_{h(n)}\} \tag{4}$$

be the collection of all subsets of A which have cardinality $[n/2]$. For $1 \leq i \leq n$, let

$$H_i = \{s \epsilon S^* : i \epsilon s\},$$

and put $\mathscr{H} = \{H_1, \ldots, H_n\}$. The possibility

$$r \epsilon \cap \{H_i : s \epsilon H_i\}, (r \epsilon S^* - \{s\}),$$

is ruled out because r cannot be a subset of s, so that some $i \epsilon A$ must satisfy $i \epsilon s - r$ and thus $s \epsilon H_i$, $r \epsilon S^* - H_i$. It follows that

$$\{s\} = \cap \{H_i : s \epsilon H_i\},$$

i.e., each element of S^* is distinguished by $\mathscr{H}$. This implies $h(n) \leq g(n)$.

The proof of (1) will be completed by showing that $g(n) \leq h(n)$. Let $\mathscr{F} = \{F_1, \ldots, F_n\}$ be a family of

finite sets with union S. For each $s \epsilon S$, let

$$F^{(s)} = \{F_i : s \epsilon F_i\},\ F_s = \cap \{F_i : s \epsilon F_i\},\ T = \{s \epsilon S : F_s = \{s\}\}.$$

Then T consists of those elements of S which are distinguished by $\mathcal{F}$ so that $|T| \leq h(n)$ is what must be proved.

A collection of sets will be called *independent* if no set-inclusions hold between any pair of members.

For example, the collection $\{F^{(s)} : s \epsilon T\}$ is an independent collection of subfamilies of an n member family. Since this collection has $|T|$ members, it suffices to show that any independent family of subsets of an n element set has at most $h(n)$ members. This can be shown using the well-known SDR theorem but we find it as easy to employ an elementary argument.

For an n element set A, let S_i denote the family of subsets of A which have cardinality i, $0 \leq i \leq n$. Each S_i, and in particular $S_{[n/2]} = S^*$, is an independent family. If $\{n/2\}$ is the smallest integer not less than $n/2$, then

$$|S_{[n/2]}| = |S_{\{n/2\}}| = h(n).$$

We shall show that any other independent family P of subsets of A can be mapped $1-1$ into $S_{[n/2]}$ and thus conclude that

$$|P| \leq |S_{[n/2]}|. \tag{5}$$

Suppose some member of P has cardinality less than $\{n/2\}$. Let P_j be the family of members of P which have smallest cardinality, say j. Let M be the family of members of S_{j+1} which contain a member of P_j. Since P is independent, $P' = M \cup (P - P_j)$ is also independent, and $P \cap M = \phi$. We will show below that

$$j < \left\{\frac{n}{2}\right\} \text{ implies } |P_j| \leq |M|, \tag{6}$$

and so $|P| \leq |P'|$.

Then by induction on the minimum cardinality of any member of P, P', etc., we obtain an independent family Q such that $|P| \leq |Q|$, $Q \cap S_i = \phi$ for $0 \leq i < \left\{\frac{n}{2}\right\}$, and $Q \cap S_i = P \cap S_i$ for $\{n/2\} < i \leq n$.

The structure of the family of all subsets of A is the same relative to the relationships "is a subset of" and "is a superset of." Hence a "mirror-image" of the preceding construction will produce from Q an independent family R such that $|Q| \leq |R|$, $R \cap S_i = \phi$ for $n \geq i > [n/2]$, and such that

$$R \cap S_i = Q \cap S_i = \phi \text{ for } [n/2] > i \geq 0.$$

In this fashion we arrive at the result $|P| \leq |R|$ and $R \subset S_{[n/2]}$.

It only remains to show (6).

Let K be the number of distinct pairs (p, m) where $p \epsilon P_j$, $m \epsilon M$, and $p \subset m$. We have

$$K = (n - j)|P_j| \tag{7}$$

since any $p \epsilon P_j$ can be extended in exactly $(n-j)$ ways to an $m \epsilon M$. Also however,

$$K \leq (j+1)|M|, \tag{8}$$

since any $m \epsilon M$ contains $j+1$ subsets of cardinality j and thus contains at most $j+1$ members of P_j.

Where $0 \leq j < \{n/2\}$,

$$(j+1)/(n-j) \leq 1$$

and therefore combining (7) and (8) we have (6), and the proof is complete.

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E-Transforms*

F. M. Ragab[1]

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The following transform pair is established:

$$g(x)=\int_0^\infty (xy)^k\left[Ep;\ \alpha_r;\ q;\ \rho_i;\ (xy)^{\pm n}\right]f(y)dy;$$

$$f(x)=n^2\int_0^\infty (xy)^{-k}E\left[\begin{matrix}0 & \\ & q;\ 1-\rho_i\mp\frac{1}{n};\ (xy)^{\pm n}\\ 1;\mp\frac{1}{n} & \\ & p+1;\ 1-\alpha_r\mp\frac{1}{n},\ 1\end{matrix}\right]g(y)dy,$$

where n is any positive integer and E is MacRobert's function and the generalized MacRobert's function, respectively.

Special choices of the parameters in the last transform lead in turn to the derivation of Hankel transform, Y-transform, K-transform, Fourier transform, Laplace transform and other integral transforms with tables to illustrate these new transforms.

Key Words: Fourier-transform, functional transforms, generalized MacRobert's function, Hankel transform, K-transform, Laplace transform, MacRobert's function, Y-transform.

1. Introduction

In this paper we discuss a new class of integral transforms and their inversion formulas. The kernel in the transform is MacRobert's E-function whose properties and definitions are to be found in [1][2] pp. 348–358, and which will be discussed further later on. In the inversion formula, the kernel is likewise the generalized E-function of MacRobert (see [1], p. 419). Known special cases of our results are the Hankel transform pair ([2], p. 3):

$$g(x)=\int_0^\infty J_\nu(xy)(xy)^{1/2}f(y)dy \tag{1}$$

$$f(x)=\int_0^\infty J_\nu(xy)(xy)^{1/2}g(y)dy; \tag{2}$$

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[1]Present address: Department of Mathematics, Faculty of Engineering, Cairo University, Cairo, Egypt, United Arab Republic.

the Y-transform pair ([2]; p. 93):

$$g(x)=\int_0^\infty Y_\nu(xy)(xy)^{1/2}f(y)dy, \tag{3}$$

$$f(x)=\int_0^\infty H_\nu(xy)(xy)^{1/2}g(y)dy; \tag{4}$$

and the K-transform pairs ([2], p. 121):

$$g(x)=\int_0^\infty K_\nu(xy)(xy)^{1/2}f(y)dy, \tag{5}$$

$$f(x)=\frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} I_\nu(xy)(xy)^{1/2}g(y)dy. \tag{6}$$

When $\nu=\pm\frac{1}{2}$, the Hankel transform and the K-transform reduce to the Fourier and Laplace transform, respectively. So it is not necessary to reproduce them here as particular cases of our results.

Section 2 contains a treatment of the E-function and preliminary results, and section 3 contains the derivation of our main results. Section 4 contains the derivation of the K-transform pair ((5) and (6)) from the Hankel transform pair ((1) and (2)); while several other particular cases are presented in section 5.

It may be noted that the parameters and the variables are such that the functions involved exist and the integrals are convergent.

2. Subsidiary Formulas

The Mellin transform pair ([3], p. 7):

$$g(s)=\int_0^\infty x^{s-1}f(x)dx, \tag{7}$$

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}g(s)ds, \tag{8}$$

will be utilized.

The E-function is defined by the Mellin-Barnes contour integral (see [1], p. 374).

$$E(p;\ \alpha_r : q;\ \rho_t : z)=\frac{1}{2\pi i}\int \Gamma(\xi)\frac{\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)}{\prod_{t=1}^{q}\Gamma(\rho_t-\xi)}z^\xi d\xi, \tag{9}$$

where the integral is taken along the η-axis with loops, if necessary, to ensure that the pole at the origin lies to the left and the poles at $\alpha_1, \alpha_2, \ldots, \alpha_p$ lie to the right of the contour. Zero and negative values of the α's and ρ's are excluded. When $p<q+1$ the contour is bent to the left at both ends. Convergence is secured if $|\arg z|<\frac{1}{2}(p-q+1)$ if $p>q+1$ and $|z|>1$ if $p=q+1$.

The generalized E-function is defined by (see [1], p. 419)

$$E\left(\begin{matrix} p;\ \alpha_r \\ q;\ \rho_s \end{matrix}\Bigg/\begin{matrix} m;\ \rho_{q+s} \\ l+1;\ \alpha_{p+r},\ 1 \end{matrix} : z\right)=\frac{1}{2\pi i}\int \frac{\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)\prod_{s=1}^{m}\Gamma(\xi-\rho_{q+s}+1)}{\prod_{s=1}^{q}\Gamma(\rho_s-\xi)\prod_{r=1}^{l}\Gamma(\xi-\alpha_{p+r}+1)}z^{\xi}d\xi$$

$$=\pi^{m-l-1}\sum_{s=1}^{m}\frac{\prod_{r=l}^{l}\sin(\rho_{q+s}-\alpha_{p+r})\pi}{\prod_{t=1}^{m}{}'\sin(\rho_{q+s}-\rho_{q+t})\pi}\,z^{\rho_{q+s}-1}$$

$$\times E\left(\begin{matrix} p+l;\ \alpha_r-\rho_{q+s}+1 \\ \rho_1-\rho_{q+s}+1,\ \ldots\ {}^{*}\ \ldots,\ \rho_{q+m}-\rho_{q+s}+1 \end{matrix} : \omega z\right), \tag{10}$$

where l, m are the integers, ω is equal to $e^{\pm i\pi}$ or 1 according as $l+m$ is even or odd and the contour passes along the η-axis from $-\infty$ to ∞ with loops, if necessary, to ensure that the poles of the integrand at the origin and at $\rho_{q+1}-1,\ \rho_{q+2}-1,\ \ldots,\ \rho_{q+m}-1$ lie to the left and the poles at $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_p$ to the right of the contour, when necessary the contour is bent to the left or to the right at both ends until it is parallel to the ξ-axis. The E-function is also defined thus:

If $p \leqslant q$; then

$$E(p;\ \alpha_r : q;\ \rho_s : z)=\frac{\Gamma(\alpha_1)\ \ldots\ \Gamma(\alpha_p)}{\Gamma(\rho_1)\ \ldots\ \Gamma(\rho_q)}pFq\left(\begin{matrix} p;\ \alpha_r; \\ q;\ \rho_s \end{matrix} -\frac{1}{z}\right), \tag{11}$$

while if $p \geqslant q+1$; then

$$E(p;\ \alpha_r : q;\ \rho_s : z)=\sum_{r=1}^{p}\prod_{s=1}^{p}{}'\Gamma(\alpha_s-\alpha_r)\left\{\prod_{t=1}^{q}\Gamma(\rho_t-\alpha_r)\right\}^{-1}\Gamma(\alpha_r)z^{\alpha_r}$$

$$F\left(\begin{matrix} \alpha_r,\ \alpha_r-\rho_1+1,\ \ldots,\ \alpha_r-\rho_q+1;\ (-1)^{p-q} \\ \alpha_r-\alpha_1+1,\ \ldots\ {}^{*}\ \ldots,\ \alpha_r-\alpha_p+1 \end{matrix}\right). \tag{12}$$

From (11) and (12) it is clear that the E-function is immediately related to the generalized hypergeometric function and reduces to simple expressions in the ordinary or Gauss hypergeometric function when $p=2$, $q=1$. For $p=q=1$ it is also evident that the E-function reduces to the confluent hypergeometric function or Kummer's function. The case $p=2$, $q=0$ yields the relations (see [1] p. 351)

$$\cos n\pi E(\tfrac{1}{2}+n,\ \tfrac{1}{2}-n : 2z)=\sqrt{(2\pi z)}e^{z}K_n(z), \tag{13}$$

and

$$E(\tfrac{1}{2}-k+m,\ \tfrac{1}{2}-k-m :: z)=\Gamma(\tfrac{1}{2}-k-m)\Gamma(\tfrac{1}{2}-k+m)z^{-k}e^{z/2}W_{k,\,m}(z), \tag{14}$$

where $K_n(z)$ and $W_{k,\,m}(z)$ are the modified Bessel function of the second kind and Whittaker function, respectively. When $p=q=0$ the E-function is just $e^{-1/z}$. When $p=0$, $q=1$, then we obtain the

Bessel function of the first kind. Thus we have

$$E(:\nu+1:x)=x^{\nu/2}J_{\nu}(2x^{-1/2}). \tag{15}$$

The case $p=1$, $q=0$ gives

$$E(\alpha::x)=\Gamma(\alpha)\left(1+\frac{1}{x}\right)^{-\alpha}. \tag{16}$$

More parameters in the E-function lead to the equivalence of the E-function with products of Hankel functions, with Lommel functions, Bessel functions and with product of Whittaker functions. The following are some examples:

$$x^{\mu}H_{\nu}^{(1)}(x)H_{\nu}^{(2)}(x)=2\,(\cos\,\nu\pi)\pi^{-3/2}x^{\mu-1}E(\tfrac{1}{2}+\nu,\ \tfrac{1}{2}-\nu,\ \tfrac{1}{2}::x), \tag{17}$$

$$S_{\mu,\,\nu}(x)=2^{\mu-1}\left\{\Gamma\left(\frac{1}{2}-\frac{\mu}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\mu}{2}+\frac{\nu}{2}\right)\right\}^{-1}\left(\frac{x}{2}\right)^{\mu-1}E\left(1,\frac{1}{2}-\frac{\mu}{2}+\frac{\nu}{2},\frac{1}{2}-\frac{\mu}{2}-\frac{\nu}{2}::\frac{x^2}{4}\right) \tag{18}$$

$$W_{k,\,m}(2ix)W_{k,\,m}(-2ix)=\pi^{-1/2}\left(\frac{x}{2}\right)^{2k}\left\{\Gamma\left(\frac{1}{2}-k+m\right)\Gamma\left(\frac{1}{2}-k-m\right)\right\}^{-1}$$

$$\times E\left(\frac{1}{2}-k+m,\frac{1}{2}-k-m,\frac{1}{2}-k,\,1-k:1-2k:\frac{x^2}{4}\right), \tag{19}$$

$$J_{\nu}(x)J_{-\nu}(x)=\pi^{-1/2}E\left(\frac{1}{2}:1-\nu,\,1+\nu:\frac{1}{x^2}\right), \tag{20}$$

$$J_{\nu}^{2}(x)=\pi^{-1/2}E\left(\frac{1}{2}+\nu:1+\nu,\,1+2\nu:\frac{1}{x^2}\right), \tag{21}$$

$$H_{\nu}(x)=E\left(\begin{matrix}1;\frac{1}{2}+\frac{1}{2}\nu\\0;\ \ldots\end{matrix}\middle/\begin{matrix}1;\frac{1}{2}+\frac{1}{2}\nu\\3;\frac{1}{2}\nu,-\frac{1}{2}\nu,1\end{matrix}:\frac{x^2}{4}\right)=\left(\frac{x^2}{4}\right)^{\frac{\nu+1}{2}}E\left(\begin{matrix}1:\frac{4}{x^2}\\\frac{3}{2},\frac{3}{2}+\nu\end{matrix}\right) \tag{22}$$

and

$$Y_{\nu}(x)=E\left(\begin{matrix}0;\ \ldots\\1;\frac{3}{2}+\frac{\nu}{2}\end{matrix}\middle/\begin{matrix}2;\,1+\frac{\nu}{2},\,1-\frac{\nu}{2}:\frac{4}{x^2}\\2,\,\frac{1}{2}\nu+\frac{3}{2},\,1\end{matrix}\right). \tag{23}$$

It is easily seen from (9) and (10) that the generalized E-function reduces to the ordinary E-function when $l=0$, $m=1$ and $\rho_{q+1}=1$.

3. The Main Results

The main theorem to be proved is

If n is any positive integer, $q \neq 0$ and if

$$g(x) = \int_0^\infty (xy)^k E[p;\ \alpha_r : q;\ \rho_t : (xy)^{\pm n}] f(y)\,dy, \tag{24}$$

then

$$f(x) = n^2 \int_0^\infty (xy)^{-k} E\left[\begin{matrix} 0; \\ 1; \mp\frac{1}{n} \end{matrix} \middle/ \begin{matrix} q;\ 1-\rho_t \mp \frac{1}{n} : (xy)^{\pm n} \\ p+1;\ 1-\alpha_r \mp \frac{1}{n},\ 1 \end{matrix}\right] g(y)\,dy, \tag{25}$$

where the kernel in the inversion formula (25) is given by (10) and the integrals are convergent.

We wish to solve the integral equation

$$g(x) = \int_0^\infty (xy)^k E[p;\ \alpha_r : q;\ \rho_s : (xy)^n] f(y)\,dy. \tag{26}$$

Using the notation of [3], p. 315, we have

$$G(s) = \int_0^\infty x^{s-1} g(x)\,dx,$$

or

$$G(1-s) = \int_0^\infty x^{-s} g(x)\,dx. \tag{27}$$

Also

$$\mathscr{R}(s) = \int_0^\infty u^{s+k-1} E[p;\ \alpha_r : q;\ \rho_t : u^n]\,du = \frac{1}{n}\int_0^\infty y^{\frac{s}{n}+\frac{k}{n}-1} E[p;\ \alpha_r : q;\ \rho_t : y]\,dy$$

$$= \Gamma\left(-\frac{s}{n}-\frac{k}{n}\right) \frac{\prod_{r=1}^{p} \Gamma\left(\alpha_r + \frac{k}{n} + \frac{s}{n}\right)}{\prod_{t=1}^{q} \Gamma\left(\rho_t + \frac{k}{n} + \frac{s}{n}\right)},$$

by (7), (8) and (9). Thus

$$n\mathscr{R}(1-s) = \Gamma\left(-\frac{1}{n}-\frac{k}{n}+\frac{s}{n}\right) \frac{\prod_{r=1}^{p} \Gamma\left(\alpha_r + \frac{k}{n} + \frac{1}{n} - \frac{s}{n}\right)}{\prod_{t=1}^{q} \Gamma\left(\rho_t + \frac{k}{n} + \frac{1}{n} - \frac{s}{n}\right)}, \tag{28}$$

and so by [3], p. 316, (27) and (28)

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{G(1-s)}{\mathscr{B}(1-s)}x^{-s}ds=\frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\prod_{l=1}^{q}\Gamma\left(\rho_l+\frac{k}{n}+\frac{1}{n}-\frac{s}{n}\right)}{\prod_{l=1}^{p}\Gamma\left(\alpha_r+\frac{k}{n}+\frac{1}{n}-\frac{s}{n}\right)}\cdot\frac{x^{-s}}{\Gamma\left(-\frac{1}{n}-\frac{k}{n}+\frac{s}{n}\right)}\times\int_0^\infty y^{-s}g(y)dyds$$

$$=n\int_0^\infty g(y)dy\int_{c-i\infty}^{c+i\infty}\frac{\prod_{l=1}^{q}\Gamma\left(\rho_l+\frac{k}{n}+\frac{1}{n}-\frac{s}{n}\right)}{\Gamma\left(-\frac{1}{n}-\frac{k}{n}+\frac{s}{n}\right)\prod_{l=1}^{p}\Gamma\left(\alpha_r+\frac{k}{n}+\frac{1}{n}-\frac{s}{n}\right)}(xy)^{-s}ds$$

$$=n^2\int_0^\infty (xy)^{-k}g(y)dy\int_{c-i\infty}^{c+i\infty}\frac{\prod_{l=1}^{q}\Gamma\left(\rho_l+\frac{1}{n}-\xi\right)}{\Gamma\left(-\frac{1}{n}+\xi\right)\prod_{l=1}^{p}\Gamma\left(\alpha_r+\frac{1}{n}-\xi\right)}(xy)^{-n\xi}d\xi$$

$$=n^2\int_0^\infty (xy)^{-k}g(y)dy\int_{c-i\infty}^{c+i\infty}\frac{\prod_{l=1}^{q}\Gamma\left(\rho_l+\frac{1}{n}+\xi\right)}{\Gamma\left(-\frac{1}{n}-\xi\right)\prod_{r=1}^{p}\Gamma\left(\alpha_r+\frac{1}{n}+\xi\right)}(xy)^{n\xi}d\xi$$

$$=n^2\int_0^\infty (xy)^{-k}g(y)E\left[\begin{matrix}0;\\ 1;-\frac{1}{n}\end{matrix}\middle/\begin{matrix}q;\ 1-\rho_l-\frac{1}{n}\\ p+1;\ 1-\alpha_r-\frac{1}{n},\ 1\end{matrix}\right]:(xy)^n\quad dy$$

by (10).

Thus the first part of the main transform pair (24) and (25) is proved. The case with $-n$ instead of n can be dealt in a similar way.

4. Derivations of Hankel and K Transforms

When $n=2$, the transform pair (24), (25) becomes

$$g(x)=\int_0^\infty (xy)^k E(p;\ \alpha_r:q;\ \rho_s:x^{-2}y^{-2})f(y)dy, \tag{29}$$

$$f(x)=4\int_0^\infty (xy)^{-k}E\left(\begin{matrix}q;\ 1-\rho_l+\frac{1}{2}:(xy)^2\\ 1;\frac{1}{2}p+1;\ 1-\alpha_r+\frac{1}{2},\ 1\end{matrix}\right)g(y)dy. \tag{30}$$

In (29) and (30), take $p=0$, $q=1$, with $\rho_1=\nu+1$, apply (15) and so obtain a generalization of Hankel transform namely:

$$g(x)=\int_0^\infty (xy)^{k-\nu}J_\nu(xy)f(y)dy, \tag{31}$$

$$f(x)=\int_0^\infty (xy)^{-k+\nu+1}J_\nu(xy)g(y)dy. \tag{32}$$

When $k=\nu+\frac{1}{2}$, then (31) and (32) reduce to the Hankel transform pair (1) and (2).

To derive the K-transform (5) and (6), we have the integral equation

$$g(y)=\int_0^\infty K_\nu(xy)(xy)^{1/2}f(x)dx.$$

Thus

$$e^{-\nu\pi i}g(y)=\int_0^\infty e^{-\nu\pi i}K_\nu(xy)(xy)^{1/2}f(x)dx,$$

$$g(ye^{\pi i})=\int_0^\infty K_\nu(xye^{\pi i})e^{\frac{i\pi}{2}}f(x)dx.$$

Subtract the last two equations using the relation

$$\pi i I_\nu(z)=e^{-\nu\pi i}K_\nu(z)-K_\nu(ze^{\pi i}),$$

so getting

$$\int_0^\infty I_\nu(xy)(xy)^{1/2}f(x)dx=e^{-\frac{\pi i}{2}}\frac{(e^{-\nu\pi i}g(y)-g(ye^{\pi i}))}{\pi i}. \tag{33}$$

In (33) write $e^{\frac{1}{2}\pi i}y$ for y and get after using the relation

$$I_\nu(ze^{\pm\frac{1}{2}\pi i})=e^{\pm\frac{1}{2}\nu\pi i}J_\nu(z),$$

$$e^{\frac{\pi i}{4}-\frac{\nu\pi i}{2}}g(ye^{-\frac{\pi i}{2}})-e^{-\frac{\pi i}{4}+\frac{\nu\pi i}{2}}g(ye^{\frac{\pi i}{2}})=\pi i\int_0^\infty J_\nu(xy)(xy)^{1/2}f(x)dx. \tag{34}$$

Apply Hankel transform to the last equation and get

$$\pi i f(x)=\int_0^\infty J_\nu(xy)(xy)^{1/2}\{e^{\frac{\pi i}{4}-\frac{\nu\pi i}{2}}g(ye^{-\frac{\pi i}{2}})-e^{-\frac{\pi i}{4}+\frac{\nu\pi i}{2}}g(ye^{\frac{\pi i}{2}})\}dy.$$

Here write $e^{-\frac{\pi i}{2}}y$ for y, apply (34) and get

$$\begin{aligned}\pi i f(x)&=\int_0^{i\infty} I_\nu(xy)(xy)^{1/2}\{e^{-\frac{\pi i}{2}-\nu\pi i}g(ye^{-\frac{i\pi}{2}})+g(y)\}dy\\&=\int_{-i\infty}^0 I_\nu(xy)(xy)^{1/2}g(y)dy+\int_0^{i\infty} I_\nu(xy)(xy)^{1/2}g(y)dy\\&=\int_{-i\infty}^{+i\infty} I_\nu(xy)(xy)^{1/2}g(y)dy\\&=\int_{c-i\infty}^{c+i\infty} I_\nu(xy)(xy)^{1/2}g(y)dy,\end{aligned}$$

provided that there are no poles of the integrand between the two contours. Thus

$$f(x)=\frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} I_\nu(xy)\,(xy)^{1/2}g(y)dy,$$

and the K-transform pair is obtained.

To derive the Y-transform, take in (24) $n=-2$, $p=1$, $q=2$ with $\alpha_1=1$, $\rho_1=\frac{3}{2}$, $\rho_2=\frac{3}{2}+\nu$, apply (22), (23) and so obtain a generalization of the Y-transform pair namely:

$$g(x)=\int_0^\infty (xy)^{k-\nu-1}H_\nu(x)f(y)dy, \tag{35}$$

then

$$f(x)=\int_0^\infty (xy)^{-k+\nu+2}Y_\nu(x)g(y)dy, \tag{36}$$

which reduces to the Y-transform pair when $k=\nu+\frac{3}{2}$.

5. Applications: Generalized Hankel Transform

In (24) and (25) write $n=2m$ $(m=1, 2, 3, \ldots)$, take the $(-)$ sign and apply (16) so obtaining another generalization of the Hankel transform pair namely

$$g(x)=\int_0^\infty (xy)^{k-\nu m}J_\nu[2x^m y^m]f(y)dy, \tag{37}$$

$$f(x)=4m^2\int_0^\infty (xy)^{m\nu+2m-1-k}J_\nu[2x^m y^m]g(y)dy, \tag{38}$$

where m is any positive integer and the integrals are convergent. When $m=1$, $k=\nu+\frac{1}{2}$ it reduces to Hankel transform pair (1) and (2). When $k=m\nu+m-\frac{1}{2}$, the last transform pair becomes:

$$g(x)=\int_0^\infty (xy)^{m-\frac{1}{2}}J_\nu[2x^m y^m]f(y)dy, \tag{39}$$

$$f(x)=4m^2\int_0^\infty (xy)^{m-\frac{1}{2}}J_\nu[2x^m y^m]g(y)dy, \tag{40}$$

where m is any positive integer. It may be noted that (39) and (40) can be deduced from (1) and (2) by a change of variable.

If $k=0$, then (31) and (32) give another form of Hankel transform namely:

$$g(x)=x^{\frac{\nu}{2}}\int_0^\infty y^{-\frac{\nu}{2}}J_\nu(2\sqrt{xy})f(y)dy, \tag{41}$$

$$f(x)=x^{\frac{\nu}{2}}\int_0^\infty y^{-\frac{\nu}{2}}J_\nu(2\sqrt{xy})g(y)dy. \tag{42}$$

Again Hankel transform pair (1) and (2) can be written in the form

$$g(x)=\int_0^\infty yJ_\nu(xy)f(y)dy. \tag{43}$$

$$f(x)=\int_0^\infty yJ_\nu(xy)g(y)dy. \tag{44}$$

Equations (43) and (44) can be extended to the case for n variables thus

$$g(x_1, x_2, \ldots, x_n)=\int_0^\infty \ldots \int_0^\infty \left[\prod_{r=1}^{n} y_r J_{\nu_r}(x_r y_r)\right] f(y_1, y_2, \ldots, y_n)dy_1dy_2 \ldots dy_n. \tag{45}$$

$$f(x_1, x_2, \ldots, x_n)=\int_0^\infty \ldots \int_0^\infty \left[\prod_{r=1}^{n} y_r J_{\nu_r}(x_r y_r)\right] g(y_1, y_2, \ldots, y_n)dy_1dy_2 \ldots dy_n. \tag{46}$$

Other particular cases of the transform pair (24) and (25) are the following new-integral transforms:

EXAMPLE 1. *The KJ-transforms:* From (24), (25) and the formula

$$E\left[\begin{matrix}0: \\ \\ 1: \frac{1}{4}\end{matrix}\middle/\begin{matrix}3: \frac{1}{4}-\nu, \frac{1}{4}-\frac{1}{2}\nu, -\frac{1}{4}-\frac{1}{2}\nu: \frac{1}{x4} \\ \\ 1: 1\end{matrix}\right] = (2\pi)^{1/2}x^{2\nu+3}K_\nu(2\sqrt{2}x)J_\nu(2\sqrt{2}x): \tag{47}$$

we arrive at the transform-pair:

$$g(x)=\frac{1}{\Gamma(1+\nu)\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\nu\right)}\int_0^\infty (xy)^k {}_0F_3\left(1+\nu, 1+\frac{1}{2}\nu, \frac{3}{2}+\frac{1}{2}\nu: -x^4y^4\right) f(y)dy. \tag{48}$$

$$f(x)=16\sqrt{2\pi}\int_0^\infty (xy)^{-k+2\nu+3}K_\nu(2\sqrt{2}xy)J_\nu(2\sqrt{2}xy)g(y)dy. \tag{49}$$

The following table illustrates some examples of the last transformation.

TABLE 1

$f(x)$	$g(x)$
e^{-x}	$\dfrac{\Gamma(k+1)}{\Gamma(1+\nu)\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\nu\right)}x^k \cdot {}_4F_3\left(\begin{matrix}\frac{k+1}{4}, \frac{k+2}{4}, \frac{k+3}{4}, \frac{k+4}{4}: \\ \\ 1+\nu, 1+\frac{1}{2}\nu, \frac{3}{2}+\frac{1}{2}\nu\end{matrix}\middle| -4^4x^4\right)$
$R(k)>-1$	$R(\nu)>-1, \quad R(k-4\nu)>0, \quad R(k-2\nu-2)>0.$

TABLE 1—Continued

$f(x)$	$g(x)$
$K_n(x)$	$2^{k-1}\dfrac{\Gamma\left(\frac{k+n+1}{4}\right)\Gamma\left(\frac{k+n+3}{4}\right)\Gamma\left(\frac{k-n+1}{4}\right)\Gamma\left(\frac{k-n+3}{4}\right)}{(2\pi)\Gamma(1+\nu)\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\nu\right)}x^k$ $\cdot {}_4F_3\left(\begin{matrix}\frac{k+n+1}{4},\ \frac{k+n+3}{4},\ \frac{k-n+1}{4},\ \frac{k-n+3}{4}; & -4^4x^4\\ 1+\nu,\ 1+\frac{1}{2}\nu,\ \frac{3}{2}+\frac{1}{2}\nu & \end{matrix}\right)$
$R(k\pm n+1)>0$	$R(\nu)>-1,\quad R(k\pm n-4\nu)>0,\quad R(k\pm n-2\nu-2)>0.$
$e^{-x}E(\gamma,\ \delta::x)$	$\dfrac{\Gamma(\gamma)\Gamma(\delta)\Gamma(k+\gamma+1)\Gamma(k+\delta+1)}{\Gamma(k+\gamma+\delta+1)\Gamma(1+\nu)\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\nu\right)}x^k$ $\cdot {}_8F_7\left(\begin{matrix}\frac{k+\gamma+1}{4},\ \frac{k+\gamma+2}{4},\ \frac{k+\gamma+3}{4},\ \frac{k+\gamma+4}{4},\\ \frac{k+\gamma+\delta+1}{4},\ \frac{k+\gamma+\delta+2}{4},\ \frac{k+\gamma+\delta+3}{4},\\ \frac{k+\delta+1}{4},\ \frac{k+\delta+2}{4},\ \frac{k+\delta+3}{4},\ \frac{k+\delta+4}{4}; & -4^4x^4\\ \frac{k+\gamma+\delta+4}{4},\ 1+\nu,\ 1+\frac{1}{2}\nu,\ \frac{3}{2}+\frac{1}{2}\nu & \end{matrix}\right)$
$R(k+\gamma+1)>0,\ R(k+\delta+1)>0$	$R(\nu)>-1,\quad R(k+\gamma-4\nu)>0,\quad R(k+\delta-4\nu)>0,$ $R(k+\gamma-2\nu-2)>0,\quad R(k+\delta-2\nu-2)>0.$
$\dfrac{\sqrt{\pi x}}{\cos n\pi}e^{-\frac{x}{2}}\Lambda_n\left(\frac{x}{2}\right)$	$\dfrac{\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(k+n+\frac{3}{2}\right)\Gamma\left(k-n+\frac{3}{2}\right)}{\Gamma(k+2)\Gamma(1+\nu)\Gamma\left(\frac{1}{2}\nu+1\right)\Gamma\left(\frac{1}{2}\nu+\frac{3}{2}\right)}x^k$ $\cdot {}_8F_7\left(\begin{matrix}\frac{2k+2n+3}{8},\ \frac{2k+2n+5}{8},\ \frac{2k+2n+7}{8},\ \frac{2k+2n+9}{8},\\ \frac{k+2}{4},\ \frac{k+3}{4},\ \frac{k+4}{4},\ \frac{k+5}{4},\ \frac{\nu+3}{2},\\ \frac{2k-2n+3}{8},\ \frac{2k-2n+5}{8},\ \frac{2k-2n+7}{8},\ \frac{2k-2n+9}{8}; & -4^4x^4\\ 1+\nu,\ 1+\frac{1}{2}\nu & \end{matrix}\right)$ $R(\nu)>-1,\ R\left(k+\frac{1}{2}\pm n-4\nu\right)>0,$
$R\left(k+\frac{3}{2}\pm n\right)>0$	$R\left(k-\frac{3}{2}\pm n-2\nu\right)>0.$

TABLE 1—Continued

$f(x)$	$g(x)$
$\Gamma\left(\frac{1}{2}-k'+m\right)\Gamma\left(\frac{1}{2}-k'-m\right)$ $x^{-k'}e^{-\frac{x}{2}}W_{k',m}(x)$	$\frac{\Gamma\left(\frac{3}{2}+k-k'+m\right)\Gamma\left(\frac{3}{2}+k-k'-m\right)}{\Gamma(k+2-2k')\Gamma(1+\nu)\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\nu\right)}x^k$ ${}_8F_7\left(\begin{matrix}\frac{2k-2k'+2m+3}{8}, \frac{2k-2k'+2m+5}{8}, \frac{2k-2k'+2m+7}{8}, \frac{2k+2k'+2m+9}{8}, \\ \frac{k-2k'+2}{4}, \frac{k-2k'+3}{4}, \frac{k-2k'+4}{4}, \frac{k-2k'+5}{4}, \\ \frac{2k-2k'-2m+3}{8}, \frac{2k-2k'-2m+5}{8}, \frac{2k-2k'-2m+7}{8}, \frac{2k-2k'-2m+9}{8}; -4^4x^4\end{matrix}\right)$ $1+\nu, 1+\frac{1}{2}\nu, \frac{3}{2}+\frac{1}{2}\nu$ $R(\nu)>-1, R\left(\frac{1}{2}+k-k'\pm m-4\nu\right)>0,$
$R\left(\frac{3}{2}+k-k'\pm m\right)>0$	$R\left(k-k'\pm m-2\nu-\frac{3}{2}\right)>0.$
$E\left(-\frac{k+1}{4}+\nu+1, -\frac{k+1}{4}+\frac{\nu}{2}+1, -\frac{k+1}{4}+\frac{\nu}{2}+\frac{3}{2}:: x^4\right)$	$2^{-1}x^{\frac{k}{2}-\frac{1}{2}}K_{\frac{k+1}{4}}(2x^2);$ $R(\nu)>-1, R(7+8\nu\pm k)>0$
$R(\nu)>-1, R(k)<-1$	$R(7+4\nu\pm k)>0.$
$x^{4\nu-k+3}(1+x^4)^{-\frac{3}{2}-\nu}$	$\frac{1}{4}x^{k-\nu}J_\nu(4x)$
$R(\nu)>-1$	$R(\nu)>-1.$

EXAMPLE 2. *The $J_\nu Y_\nu$-transform:* From (24), (25) and the formula

$$x^\nu J_\nu(\sqrt{x})Y_\nu(\sqrt{x})=-\pi^{-1/2}E\left(\begin{matrix}0 \\ 1; 1\end{matrix}\middle|\begin{matrix}2; 1-2\nu, 1-\nu; \frac{1}{x} \\ 2; \frac{1}{2}-\nu, 1\end{matrix}\right); \tag{50}$$

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we arrive at the transform pair

$$g(x)=\frac{\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(1+2\nu)\Gamma(1+\nu)}\int_0^\infty (xy)^k {}_1F_2\left(\begin{matrix}\frac{3}{2}+\nu;\ -xy\\ 1+2\nu,\ 1+\nu\end{matrix}\right) f(y)dy, \tag{51}$$

$$f(x)=-\pi^{1/2}\int_0^\infty (xy)^{-k+\nu} J_\nu(\sqrt{x})Y_\nu(\sqrt{x})g(y)dy. \tag{52}$$

Some examples of this transform are given in the following table.

TABLE 2

$f(x)$	$g(x)$
e^{-x}	$x^k\dfrac{\Gamma(1+k)\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(1+2\nu)\Gamma(1+\nu)}$ $_2F_2\left(\begin{matrix}1+k,\ \frac{3}{2}+\nu;\ -x\\ 1+2\nu,\ 1+\nu\end{matrix}\right)$:
$R(k+1)>0$	$R(k)>\nu-\frac{1}{2}$, $\quad R(1+2\nu)>0$.
$e^{-x}E(\gamma,\ \delta;;x)$	$x^k\dfrac{\Gamma(\gamma)\Gamma(\delta)\Gamma(\gamma+k+1)\Gamma(\delta+k+1)\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(1+2\nu)\Gamma(1+\nu)\Gamma(\gamma+\delta+k+1)}$ $_3F_2\left(\begin{matrix}\gamma+k+1,\ \delta+k+1,\ \frac{3}{2}+\nu;\ -x\\ 1+2\nu,\ 1+\nu,\ \gamma+\delta+k+1\end{matrix}\right)$: $R(1+2\nu)>0$, $\quad R(\nu-\gamma-k)<\frac{1}{2}$,
$R(\gamma+k+1)>0$, $\quad R(\delta+k+1)>0$	$R(\nu-\delta-k)<\frac{1}{2}$.
$\dfrac{\sqrt{\pi x}}{\cos n\pi}e^{-\frac{x}{2}}K_n\left(\frac{x}{2}\right)$	$x^k\dfrac{\pi\Gamma\left(\frac{3}{2}+k+n\right)\Gamma\left(\frac{3}{2}+k-n\right)\Gamma\left(\frac{3}{2}+\nu\right)}{\cos n\pi\Gamma(1+2\nu)\Gamma(1+\nu)\Gamma(2+k)}$ $_3F_3\left(\begin{matrix}\frac{3}{2}+k+n,\ \frac{3}{2}+k-n,\ \frac{3}{2}+\nu;\ -x\\ 1+2\nu,\ 1+\nu,\ 2+k\end{matrix}\right)$:
$R\left(\frac{3}{2}+k\pm n\right)>0$	$R(1+2\nu)>0$, $\quad R(1+k-\nu\pm n)>0$.

TABLE 2—Continued

$f(x)$	$g(x)$
$\Gamma\left(\frac{1}{2}-k'+m\right)\Gamma\left(\frac{1}{2}-k'-m\right)x^{-k'}e^{-\frac{x}{2}}$	$x^{k}\frac{\Gamma\left(\frac{1}{2}-k'+m\right)\Gamma\left(\frac{1}{2}-k'-m\right)\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(1+2\nu)\Gamma(1+\nu)\Gamma(2-2k'+k)}$
$W_{k',m}(x)$	$\Gamma\left(\frac{3}{2}-k'+k+m\right)\Gamma\left(\frac{3}{2}-k'+k-m\right)$
	${}_3F_3\left(\begin{matrix}\frac{3}{2}-k'+k+m,\ \frac{3}{2}-k'+k-m,\ \frac{3}{2}+\nu; \\ 1+2\nu,\ 1+\nu,\ 2-2k'+k\end{matrix}\ -x\right)$;
$R\left(\frac{3}{2}+k-k'\pm m\right)>0$	$R(1+2\nu)>0,\quad R(1+k-k'-\nu\pm m)>0.$
$K_{2n}(2\sqrt{x})$	$x^{k}\frac{\Gamma(1+k+n)\Gamma(1+k-n)\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(1+2\nu)\Gamma(1+\nu)}$
	${}_3F_2\left(\begin{matrix}1+k+n,\ 1+k-n,\ \frac{3}{2}+\nu; \\ 1+2\nu,\ 1+\nu\end{matrix}\ -x\right)$;
$R(1+k\pm n)>0$	$R(1+2\nu)>0,\quad R\left(\frac{1}{2}+k-\nu\pm n\right)>0.$
$\Gamma(2\nu-k)\Gamma(\nu-k)\left\{\Gamma\left(\frac{1}{2}+\nu-k\right)\right\}^{-1}$	$2x^{\frac{k}{2}-\frac{1}{2}}K_{k+1}(2\sqrt{x});$
${}_2F_1\left(\begin{matrix}2\nu-\kappa,\ \nu-k; \\ \frac{1}{2}+\nu-k\end{matrix}\ -\frac{1}{x}\right)$	
$R\left(\nu+\frac{1}{2}-k\right)>0,\quad R(2\nu+1)>0$	$R(2\nu+1)>0,\quad R(2\nu-k+1)>0.$

EXAMPLE 3. *The* $\{J^2_{-\nu}-J^2_{\nu}\}$ *transform:* From (24), (25), and the formula

$$E\left(\begin{matrix}0 \\ 1;\ 1\end{matrix}\ \middle|\ \begin{matrix}2;\ 1+\nu,\ 1-\nu; \\ 2;\ \frac{1}{2},\ 1\end{matrix}\ \frac{1}{x}\right)=\frac{\sqrt{\pi}}{2\sin\nu\pi}\left\{J^2_{-\nu}(\sqrt{x})-J^2_{\nu}(\sqrt{x})\right\};\tag{53}$$

we arrive at the transform pair

$$g(x)=\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1+\nu)\Gamma(1-\nu)}\int_0^\infty (xy)^k {}_1F_2\left(\begin{matrix}\frac{3}{2}; -xy\\ 1+\nu,\ 1-\nu\end{matrix}\right) f(y)dy \tag{54}$$

$$f(x)=\frac{\sqrt{\pi}}{2\sin\nu\pi}\int_0^\infty (xy)^{-k}\{J_{-\nu}^2(\sqrt{x})-J_\nu^2(\sqrt{x})\}g(y)dy. \tag{55}$$

Particular cases of this transform are given in the following table.

TABLE 3

$f(x)$	$g(x)$
e^{-x};	$x^k\dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma(1+k)}{\Gamma(1+\nu)\Gamma(1-\nu)}$ ${}_2F_2\left(\begin{matrix}\frac{3}{2},\ 1+k; -x\\ 1+\nu,\ 1-\nu\end{matrix}\right)$;
$R\left(k+\frac{1}{2}\right)>0$	$R\left(k+\frac{1}{2}\right)>0, \quad R(1\pm\nu)>0$
$K_{2n}(2\sqrt{x})$;	$x^k\dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma(1+k+n)\Gamma(1+k-n)}{\Gamma(1+\nu)\Gamma(1-\nu)}$ ${}_3F_2\left(\begin{matrix}\frac{3}{2},\ 1+k+n,\ 1+k-n; -x\\ 1+\nu,\ 1-\nu\end{matrix}\right)$;
$R\left(\frac{1}{2}+k\pm n\right)>0$	$R(1\pm\nu)>0, \quad R\left(\frac{1}{2}+k\pm n\right)>0.$
$e^{-x}E(\gamma,\delta::x)$;	$x^k\dfrac{\Gamma(\gamma+k+1)\Gamma(\delta+k+1)\Gamma\left(\frac{3}{2}\right)\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+k+\delta+1)\Gamma(1+\nu)\Gamma(1-\nu)}$ ${}_3F_2\left(\begin{matrix}\frac{3}{2},\ \gamma+k+1,\ \delta+k+1; -x\\ 1+\nu,\ 1-\nu,\ \gamma+\delta+k+1\end{matrix}\right)$;
$R\left(\gamma+k+\frac{1}{2}\right)>0, R\left(\delta+k+\frac{1}{2}\right)>0$	$R(1\pm\nu)>0, \quad R\left(\gamma+k+\frac{1}{2}\right)>0, \quad R\left(\delta+k+\frac{1}{2}\right)>0.$

TABLE 3—Continued

$f(x)$	$g(x)$
$\frac{\sqrt{\pi x}}{\cos n\pi} e^{-\frac{x}{2}} K_n\left(\frac{x}{2}\right)$;	$x^k \frac{\pi\Gamma\left(\frac{3}{2}+k+n\right)\Gamma\left(\frac{3}{2}+k-n\right)\Gamma\left(\frac{3}{2}\right)}{\cos n\pi\,\Gamma(2+k)\Gamma(1+\nu)\Gamma(1-\nu)}$ ${}_3F_3\left(\begin{matrix}\frac{3}{2}, \frac{3}{2}+k+n, \frac{3}{2}+k-n; -x \\ 2+k,\ 1+\nu,\ 1-\nu\end{matrix}\right)$;
$R(1+k\pm n)>0$	$R(1\pm\nu)>0, \quad R(1+k\pm n)>0.$
$\Gamma\left(\frac{1}{2}-k'+m\right)\Gamma\left(\frac{1}{2}-k'-m\right) x^{-k'}$ $e^{-\frac{x}{2}} W_{k',m}(x)$;	$x^k \frac{\Gamma\left(\frac{1}{2}-k'+m\right)\Gamma\left(\frac{1}{2}-k'-m\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(2-2k'+k)\Gamma(1+\nu)\Gamma(1-\nu)}$ $\Gamma\left(\frac{3}{2}-k'+k+m\right)\Gamma\left(\frac{3}{2}-k'+k-m\right)$ ${}_3F_3\left(\begin{matrix}\frac{3}{2}-k'+k+m, \frac{3}{2}-k'+k-m, \frac{3}{2}; -x \\ 2-2k'+k,\ 1+\nu,\ 1-\nu\end{matrix}\right)$;
$R(1+k-k'\pm m)>0$	$R(1\pm\nu)>0, \quad R(1+k-k'\pm m)>0.$
$\frac{\Gamma(\nu-k)\Gamma(-\nu-k)}{\Gamma\left(\frac{1}{2}-k\right)}$ ${}_2F_2\left(\begin{matrix}\nu-k, -\nu-k; -\frac{1}{x} \\ \frac{1}{2}-k\end{matrix}\right)$;	$2x^{\frac{k}{2}-\frac{1}{2}} K_{k+1}(2\sqrt{x})$;
$R\left(k-\frac{1}{2}\right)<0$	$R(1\pm\nu)>0.$

6. References

[1] MacRobert, T. M., Functions of a complex variable, 5 edit. (Macmillan Co., 1962).

[2] Erdely, A., Magnus, W., Oberhetinger, F. and Tricomi, F., Tables of integral transforms, Vol. 2 (McGraw-Hill, Book Co., New York, N.Y., 1954).

[3] Titchmarsh, E. C., Introduction to the theory of Fourier integrals, Oxford (1937).

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Three Observations on Nonnegative Matrices*

A. J. Hoffman**

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Some results on nonnegative matrices are proved, of which the following is representative: Let $A=(a_{ij})$ be a nonnegative row stochastic matrix. If $\lambda \neq 1$ is an eigenvalue of A, then

$$|\lambda| \leq \min\left(1-\sum_j \min_i a_{ij}, \sum_j \max_i a_{ij}-1\right).$$

Key Words: Bounds, eigenvalues, nonnegative matrices.

1. Introduction

In this note, the following results are proved about nonnegative matrices $A=(a_{ij})$ of order n.

THEOREM A: *If* A *is symmetric, and* $c_i=\sum_j a_{ij}$, *then*

$$\sum_{i,j} (A_m)_{i,j} \leq \sum_i c_i^m, \qquad m=1, 2, \ldots .$$

This proves a conjecture of London [3], who has already proved Theorem A for small m and all n, and small n and all m.

THEOREM B: *If* A *is stochastic* $\left(i.e., \sum_j a_{ij}=1\right)$, *and* $\lambda \neq 1$ *is an eigenvalue of* A, *then*

$$|\lambda| \leq \min\left(1-\sum_j \min_i a_{ij}, \sum_j \max_i a_{ij}-1\right).$$

Goldberg ([1],[1] Theorem 1) has shown that

$$|A| \leq \left(1-\sum_j \min_i a_{ij}\right)^{n-1}.$$

Since $|A|$ is the product of the eigenvalues of A, and 1 is an eigenvalue of A, Goldberg's theorem is a consequence of our inequality.

THEOREM C: *If* A *is stochastic and nonsingular,* $B=A^{-1}$, *then*

$$\max_{i,j} |b_{ij}| = \max_{\substack{y \mid y \neq 0 \\ yA \geq 0}} \frac{\max_j |y_i|}{\sum_j y_i}.$$

Goldberg ([1], Theorem 2) has shown that $|\det A|$ is at most the reciprocal of the right-hand side of the above equation, and Theorem C is an effort to amplify Goldberg's theorem by characterizing the right-hand side in terms of elements of A^{-1}.

2. Proof of Theorem A

We denote by the cardinality of a set S by $|S|$, and the set $\{1, \ldots, n\}$ by N. For any $S \subset N$, u_S is the vector $(u_1, \ldots, u_n)$ with $u_j=1$ if $j \in S$, $u_j=0$ if $j \notin S$. A nonnegative matrix A is substochastic if $u_N'A \leq u_N'$ and $Au_N \leq u_N$. A subpermutation matrix is a substochastic matrix in which every entry is 0 or 1. If x and y are nonnegative vectors, then $x \ll y$ means

$$(2.1) \qquad \max_{|S|=k} (u_S, x) \leq \max_{|S|=k} (u_S, y) \qquad k=1, \ldots, n.$$

The following are known (see [3]).

(2.2) If x and y are nonnegative vectors, $x \ll y$ if and only if there exists a substochastic matrix A such that $x=Ay$.

(2.3) The convex hull of the subpermutation matrices is the set of substochastic matrices.

If $a=(a_1, \ldots, a_n)$ and $b=(b_1, \ldots, b_n)$ are nonnegative vectors with

(2.4) $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$, ab is the vector $(a_1b_1, \ldots, a_nb_n)$.

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**Present address: IBM Watson Research Center, York Town Heights, New York 10598.

[1] Figures in brackets indicate the literature references at the end of this paper.

Also, we define Ξ_a to be the set of nonnegative matrices A such that

$$Au_N \ll a \text{ and } (u_N'A)' \ll a. \tag{2.5}$$

Note that Ξ_{u_N} is the set of substochastic matrices.

LEMMA 2.1: *If* a *and* b *satisfy* (2.4), $X \in \Xi_a$, $Y \in \Xi_b$, *then* $XY \in \Xi_{ab}$.

PROOF: By (2.1) and (2.5), we need only show, for $S \subset N$, that

$$u_S'XYu_N \leq a_1b_1 + \ldots + a_{|S|}b_{|S|}, \tag{2.6}$$

and

$$u_N'XYu_S \leq a_1b_1 + \ldots + a_{|S|}b_{|S|}.$$

We prove only (2.6), the other inequality following by symmetry. Since X is nonnegative, $(u_S'X)' \leq (u_N'X)' \ll a$. It follows that

$$\max_{|T|=h} u_T'(u_S'X)' \leq a_1 + \ldots + a_h, \tag{2.7}$$

$$h = 1, \ldots, |S|.$$

Also, since $Xu_N \ll a$, it follows that

$$\max_{|T|=h} u_T'(u_S'X)' \leq a_1 + \ldots + a_{|S|}, \tag{2.8}$$

$$h = |S|, \ldots, n.$$

Therefore, if $\mathbf{a}^{|S|}$ *denotes the vector* $(a_1, \ldots, a_{|S|}, 0, \ldots, 0)$, *we have from* (2.7) *and* (2.8) *that*

$$(u_S'X)' \ll a^{|S|}. \tag{2.9}$$

Using (2.2) and (2.3), we see that $(u_S'X)'$ is in the convex hull of all vectors formed by applying subpermutation matrices to $a^{|S|}$. Since $Y_{u_N} \ll b$, Y_{u_N} is in the convex hull of all vectors formed by applying subpermutation matrices to b. But the left side of (2.6) is the inner product of vectors in these respective convex hulls, and is at most the maximum inner product obtained from a vertex of one hull and a vertex of the other hull. From the well-known inequality ([2], p. 268), this is at most $\sum_{i=1}^{|S|} a_ib_i$, which is (2.6).

COROLLARY: *If* $X \in \Xi_a$, $X^m \in \Xi_{a^m}$, $\quad m = 1, 2, \ldots$

Theorem A is obviously a special case of the corollary.

3. Proof of Theorem B

If $\lambda \neq 1$, and $\lambda v' = v'A$, then $(v, u_N) = 0$, since $Au_N = 1u_N$. Let $c_j = \min_i a_{ij}$. Consider the matrix $A_c = (a_{ij} - c_j)$. Since $(v, u_N) = 0$, $v'A_c = v'A$. Hence,

$$\lambda v' = v'A = v'A_c. \tag{3.1}$$

Taking absolute values in (3.1), we have

$$|\lambda|\,|v_j| \leq \sum_i |v_i|(a_{ij} - c_j). \tag{3.2}$$

But A_c is a nonnegative matrix, and it is well-known (see [5]) that if a nonnegative nonzero vector x and a nonnegative number μ satisfy

$$\mu x_j \leq \sum_i x_ib_{ij}$$

for a nonnegative matrix B, then μ is at most the largest eigenvalue of B. Applying this to (3.2), and observing that $1 - \Sigma c_j$ is the largest eigenvalue of A_c, we get the first inequality of Theorem B. The second is proved in an analogous manner.

4. Proof of Theorem C

Let y be any real vector such that $y'A \geq 0$, and $z' = y'A$. Then, with $B = A^{-1}$,

$$\max_{\substack{y \\ y \neq 0 \\ y'A \geq 0}} \frac{\max_j |y_j|}{\sum_j y_j} = \max_{\substack{z \geq 0 \\ z \neq 0}} \frac{\max_j |(z'B)_j|}{z'Bu}, \tag{4.1}$$

where $u = (1, \ldots, 1)$. But $Au = u$ implies $Bu = u$, and numerator and denominator of the right side of (4.1) are homogeneous in z. Hence,

$$\max_{\substack{z \geq 0 \\ z \neq 0}} \frac{\max_j |(z'B)_j|}{z'Bu} = \max_{\substack{z \geq 0 \\ \sum z_j = 1}} \max_j |(z'B)_j|. \tag{4.2}$$

Let i_0, j_0 be such that $|b_{i_0,j_0}| = \max_{i,j} |b_{i,j}|$. Then $|(z'B)_j| \leq \sum_i |z_ib_{ij}| \leq \sum_i z_i|b_{ij}| \leq \max_i |b_{ij}| \leq |b_{i_0,j_0}|$, since $z_i \geq 0$, $\sum z_i = 1$. Consequently, the right-hand side of (4.2) is at most $|b_{i_0,j_0}|$. But this number is achieved if z is the vector with 1 in position i_0, and is 0 everywhere else. Consequently, the right-hand side of (4.2) is $|b_{i_0,j_0}|$, which combines with (4.1) to prove Theorem C. Noting that

$$1 \geq |\text{ cofactor of } i_0, j_0 \text{ in } A| = |b_{i_0,j_0}| \cdot |\det A|,$$

one can deduce an alternative proof of [1], Theorem 2.

Incidentally, it is manifest that essentially the same arguments also prove:

$$\max_{i,j} b_{ij} = \max_{y \mid \substack{y'A \geq 0 \\ y \neq 0}} \frac{\max_j y_j}{\sum_j y_j}$$

and

$$\min_{i,j} b_{ij} = \min_{y \mid \substack{y'A \geq 0 \\ y \neq 0}} \frac{\min_j y_j}{\sum_j y_j}.$$

We thank D. London for stimulating conversations about the contents of Theorem A.

5. References

[1] K. Goldberg, Upper bounds for the determinent of a row stochastic matrix, J. Res. NBS **70B** (Math. and Math. Phys.), No. 2, 157–158 (1966).
[2] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities (Cambridge University Press, 1964).
[3] D. London, Two inequalities in nonnegative symmetric matrices, Pac. Jour. Math. **16,** 515–536 (1966).
[4] L. Mirsky, On a Convex Set of Matrices, Arch. der Math. **10,** 88–92 (1959).
[5] H. Wielandt, Uzerlegbare, nicht negative Matrizen, Math. Zeit. **52,** 642–648 (1950).

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Additional Remarks on a Theorem of M. Riesz[1]

John M. Smith

Institute for Applied Technology, National Bureau of Standards, Washington, D.C. 20234

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Let V be a real four-dimensional vector space, whose underlying geometry is the metric defined by the matrix

$$K=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$$

The following theorem is proved.

THEOREM: *If A and B are real, skew-symmetric, orthogonal (with respect to K) linear transformations on V, then $[A, B]=AB-BA$ is a multiple of a real, skew-symmetric, orthogonal (with respect to K), linear transformation on V.*

The theorem is proved by use of the first and second regular representations of the real quaternions.

Methods are given for constructing all 4×4 matrices which are skew-symmetric and orthogonal with respect to K, and all 4×4 matrices which are skew-symmetric (in the Euclidean sense) and orthogonal with respect to K.

Key Words: Commutator, matrix, orthogonal, quaternions, skew-symmetric, regular representation.

1. Introduction

M. Riesz, in a series of lectures [4][2] dealing with Clifford algebras delivered at the University of Maryland (September, 1957, to February, 1958) proved the following theorem.

THEOREM: *If* A *and* B *are real, skew-symmetric, orthogonal linear transformations on* E_4 *(a real four-dimensional vector space on which there has been imposed a Euclidean metric), then* [A, B]=AB−BA *is a multiple of a real, skew-symmetric, orthogonal transformation on* E_4.

Subsequently, M. Pearl [3] proved this theorem using only the properties of the matrices associated with these transformations and the first and second regular representations of the real quaternions. Furthermore, Pearl proved that if the underlying geometry is changed from the Euclidean metric to the Lorentzian metric then the theorem is satisfied vacuously.

The purpose of this paper is to prove the theorem in the remaining four-dimensional case. That is, when the underlying geometry of the vector space is the metric defined by the matrix

$$K=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$$

The main tools used are the first and second regular representations of the real quaternions. Methods are given for constructing all 4×4 matrices which are skew-symmetric and orthogonal with respect to K, and for constructing all 4×4 matrices which are skew-symmetric (in the Euclidean sense) and orthogonal with respect to K.

[1] This paper formed part of the author's thesis for the Master of Arts degree, University of Maryland, 1961.

[2] Figures in brackets indicate the literature references at the end of this paper.

2. Regular Representations

Consider any algebra $\mathfrak{A}$ over the field $\mathfrak{F}$. If $\mathfrak{A}$ has a basis consisting of $e_1, e_2, \ldots, e_n$ then

$$e_i e_j = \sum_{k=1}^{n} c_{ijk} e_k, \qquad c_{ijk} \epsilon \mathfrak{F} \tag{1}$$

Using the method and notation of Pearl [3], if we define the $n \times n$ matrices $R_i = (c_{isr})$ and $S_i = (c_{ris})$, where r denotes the row index and s the column index, by associativity in $\mathfrak{A}$ we have

$$R_i R_j = \sum_{k=1}^{n} c_{ijk} R_k \tag{2}$$

$$S_i S_j = \sum_{k=1}^{n} c_{ijk} S_k \tag{3}$$

and

$$R_i' S_j = S_j R_i' \tag{4}$$

where R_i' is the transpose of R_i.

If for any element a of $\mathfrak{A}$,

$$a = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n, \ \alpha_i \epsilon \mathfrak{F}$$

we define

$$R(a) = \alpha_1 R_1 + \alpha_2 R_2 + \ldots + \alpha_n R_n,$$

$$S(a) = \alpha_1 S_1 + \alpha_2 S_2 + \ldots + \alpha_n S_n \tag{5}$$

then the mappings $a \to R(a)$ and $a \to S(a)$ of $\mathfrak{A}$ into the algebras of the matrices, $\mathfrak{R}$ and $\mathfrak{S}$, are homomorphisms called the first and second regular representations of $\mathfrak{A}$. Furthermore, if $\mathfrak{A}$ has an identity element, then these mappings are isomorphisms.

In the case in which $\mathfrak{A}$ is the algebra of the real quaternions, $\mathfrak{F}$ is the real field and $\mathfrak{A}$ has basis elements e_0, e_1, e_2, e_3 where e_0 is the identity element. It follows that $R_0 = S_0 = I$, the identity matrix, and

$$R_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

Since $R_1, R_2, R_3, S_1, S_2, S_3$ clearly span the six dimensional vector space $\mathfrak{O}$ of 4×4 skew-symmetric matrices, it follows that they are linearly independent and form a basis for $\mathfrak{O}$.

Thus, every 4×4 skew-symmetric matrix Q has a unique representation as $R(a) + S(b)$ where a and b are pure quaternions. (A pure quaternion is a quaternion whose e_0 − coefficient is zero.)

3. Skew-Symmetric Matrices

Now let us consider some properties of skew-symmetric matrices, in particular, the matrices $R(a)$ and $S(b)$.

Since each of the R's and S's is skew-symmetric, (4) becomes $R(a)S(b) = S(b)R(a)$. Now consider the mapping σ of the real vector $\mathbf{a} = (\alpha_1, \alpha_2, \alpha_3)$ into the pure quaternion $a = \sigma(\mathbf{a})$. Using this, Pearl [3] proves the following:

LEMMA 1: *If* a *and* b *are orthogonal, pure quaternions* $(\mathbf{a} \cdot \mathbf{b} = 0)$, *then* R(a)R(b) = − R(b)R(a) *and* S(a)S(b) = − S(b)S(a).

For the pure quaternion $a = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ we define the norm, $N(a)$, to be $\alpha_1^2 + \alpha_2^2 + \alpha_3^2$. Clearly for any real number α, $N(\alpha a) = \alpha^2 N(a)$ and furthermore $a^2 = -N(a)e_0$. Thus

$$R(a)^2 = S(a)^2 = -N(a)I. \tag{7}$$

For the skew-symmetric matrix $Q = R(a) + S(b)$ we define the conjugate, $\bar{Q}$, to be $R(a) - S(b)$, and the norm, $N(Q)$, to be $N(b) - N(a)$. Thus,

$$Q\bar{Q} = \bar{Q}Q = R(a)^2 - S(b)^2 = N(Q)I. \tag{8}$$

Furthermore, if $\mathbf{c}$ is any vector orthogonal to $\mathbf{b}$, then by lemma 1

$$S(c)QS(c)^{-1} = S(c)[R(a) + S(b)]S(c)^{-1} = R(a) - S(b) = \bar{Q} \tag{9}$$

Thus, $\overline{Q}$ and Q are similar and hence taking determinants of both sides of (8) we have

$$\text{Det } Q \text{ Det } \overline{Q} = (\text{Det } Q)^2 = N(Q)^4$$

and since the determinant of a real skew-symmetric matrix is nonnegative, we have

LEMMA 2: *Det* $Q = Det\ \overline{Q} = N(Q)^2$.

4. K-Skew-Symmetric, K-Orthogonal Matrices

Let the metric on the real four-dimensional vector space V be defined by the matrix

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

We defined a matrix A to be orthogonal with respect to K (K-orthogonal) if $A'KA = K$, and we define a matrix B to be skew-symmetric with respect to K (K-skew-symmetric) if $x'BKx = 0$ for every vector x in V. Thus B is K-skew-symmetric if and only if BK is skew-symmetric in the Euclidean sense.

Consider now a matrix A which is both K-skew-symmetric and K-orthogonal. Since A is K-skew-symmetric there exists a skew-symmetric matrix Q such that $AK = Q$ or $A = QK$. Thus we have

$$A' = KQ' = -KAK. \tag{10}$$

Furthermore, since A is K-orthogonal, by definition $A'KA = K$ and hence by (10)

$$A^2 = -I.$$

Thus
$$A^{-1} = -A. \tag{11}$$

But also
$$A^{-1} = (QK)^{-1} = KQ^{-1}. \tag{12}$$

If we define the norm of A, $N(A)$, to be the norm of Q for $AK = Q$ we have

LEMMA 3: *If the matrix* A *is* K*-skew-symmetric and* K*-orthogonal then* N(A) $= \pm 1$.

PROOF: $A'KA = K$ and $AK = Q$. Hence $Q'KQ = K$, and taking determinants of the terms (det $Q)^2 = 1$. Thus det $Q = \pm 1$. Since Q is skew-symmetric det $Q = 1$. By lemma 2, det $Q = N(Q)^2$. Thus $N(A) = N(Q) = \pm 1$.

LEMMA 4: $KR_iK = \pm R_i$, $KS_iK = \pm S_i$, $+$ *if* i $= 1$, $-$ *if* i $= 2, 3$. Now we may prove the following

THEOREM 1: *The matrix* A *is* K*-skew-symmetric and* K*-orthogonal if and only if one of the following conditions is satisfied:*

1. N(A) $= 1$ *and* A *has the form* $\alpha_2 R_2 K + \alpha_3 R_3 K + \beta_1 S_1 K$.

2. N(A) $= -1$ *and* A *has the form* $\alpha_1 R_1 K + \beta_2 S_2 K + \beta_3 S_3 K$.

PROOF: Let A be K-skew-symmetric and K-orthogonal. Then by (11) and (12), for $A = QK$ we have

$$A^{-1} = -QK = KQ^{-1}.$$

Thus by (8)
$$-KQK = \frac{1}{N(Q)}\overline{Q}. \tag{13}$$

Let $Q = R(a) + S(b)$ where $R(a) = \alpha_1 R_1 + \alpha_2 R_2 + \alpha_3 R_3$, and $S(b) = \beta_1 S_1 + \beta_2 S_2 + \beta_3 S_3$. Then we have two cases.

Case 1. $N(A) = N(Q) = +1$.

Substituting in (13) for Q and $\overline{Q}$ and applying lemma 4 we obtain

$$-2\alpha_1 R_1 + 2\beta_2 S_2 + 2\beta_3 S_3 = 0.$$

Since R_1, S_2, and S_3 are linearly independent, $\alpha_1 = \beta_2 = \beta_3 = 0$ and hence if $N(A) = 1$, $A = \alpha_2 R_2 K + \alpha_3 R_3 K + \beta_1 S_1 K$.

Case 2. $N(A) = N(Q) = -1$.

Again, substituting in (13) for Q and $\overline{Q}$ and applying lemma 4, we obtain $\alpha_2 = \alpha_3 = \beta_1 = 0$ and hence if $N(A) = -1$, $A = \alpha_1 R_1 K + \beta_2 S_2 K + \beta_3 S_3 K$.

Conversely, if $N(A) = 1$ and $A = \alpha_2 R_2 K + \alpha_3 R_3 K + \beta_1 S_1 K$, AK is skew-symmetric, and hence A is K-skew-symmetric. Furthermore,

$$A'KA = (-KQ)(KQK) = +KQ\overline{Q} \text{ by lemma 4}$$

$$= K[N(Q)I] = K.$$

Thus A is K-orthogonal. Similarly if $N(A) = -1$ and $A = \alpha_1 R_1 K + \beta_2 S_2 K + \beta_3 S_3 K$, A is K-skew-symmetric and K-orthogonal.

Using (6) and lemma 4 we may construct the following table of commutators.

$[\,,]$	R_1K	R_2K	R_3K	S_1K	S_2K	S_3K
R_1K	0	0	0	0	$2S_3K$	$2S_2K$
R_2K	0	0	$2S_1K$	$-2R_3K$	0	0
R_3K	0	$-2S_1K$	0	$-2R_2K$	0	0
S_1K	0	$2R_3K$	$2R_2K$	0	0	0
S_2K	$-2S_3K$	0	0	0	0	$2R_1K$
S_3K	$-2S_2K$	0	0	0	$-2R_1K$	0

We can now prove the analog to Riesz' Theorem. Let A and B be two K-skew-symmetric, K-orthogonal matrices. Then we have,

Case 1. $N(A) = N(B) = 1$. Using the linearity of commutators and the results of the table,

$$[A, B]=[\alpha_2R_2K+\alpha_3R_3K+\beta_1S_1K,\ \gamma_2R_2K+\gamma_3R_3K+\delta_1S_1K]$$

$$=2(\beta_1\gamma_3-\alpha_3\delta_1)R_2K+2(\beta_1\gamma_2-\alpha_2\delta_1)R_3K+2(\alpha_2\gamma_3-\alpha_2\gamma_2)S_1K$$

which is a multiple of a K-skew-symmetric, K-orthogonal matrix with norm 1, or zero if $A=\pm B$.

Case 2. $N(A)=N(B)=-1$.

$$[A, B]=[\alpha_1R_1K+B_2S_2K+\beta_3S_3K,\ \gamma_1R_1K+\delta_2S_2K+\delta_3S_3K]=2(\beta_2\delta_3-\beta_3\delta_2)R_1K+2(\alpha_1\delta_3-\beta_3\gamma_1)S_2K+2(\alpha_1\delta_2-\beta_2\gamma_1)S_3K$$

which is a multiple of a K-skew-symmetric, K-orthogonal matrix with norm -1, or zero if $A=\pm B$.

Case 3. $N(A)=-N(B)=-1$.

$$[A, B]=[\alpha_2R_2K+\alpha_3R_3K+\beta_1S_1K,\ \gamma_1R_1K+\delta_2S_2K+\delta_3S_3K].$$

$$=0.$$

Thus the theorem is satisfied in all possible cases.

COROLLARY: *Let* A *and* B *be* K*-skew-symmetric,* K*-orthogonal matrices such that* $A\neq\pm B$. *Then* A *and* B *commute if and only if* $N(A)N(B)=-1$.

5. Skew-Symmetric, K-Orthogonal Matrices

Since we have a method for constructing all K-skew-symmetric, K-orthogonal matrices, and Pearl [3] gives a method for constructing all skew-symmetric, orthogonal matrices, the question arises, what is the form of a skew-symmetric, K-orthogonal matrix?

THEOREM 2: *The matrix* M *is skew-symmetric (in the Euclidean sense) and* K*-orthogonal if and only if* $M=AK$ *where* A *is a* K*-skew-symmetric,* K*-orthogonal matrix. (Thus by Theorem 1,* M *is either of the form*

$$\alpha_2R_2+\alpha_3R_3+\beta_1S_1,\ \beta_1^2-\alpha_2^2-\alpha_3^2=1,\ \text{or}$$

$$\alpha_1R_1+\beta_2S_2+\beta_3S_3,\ \beta_2^2+\beta_3^2-\alpha_1^2=-1.)$$

PROOF: If M is skew-symmetric and K-orthogonal, then M' and $-M$ and $M'KM=K$. Hence

$$(KM)^2=(MK)^2=-I$$

and thus $$M^{-1}=-KMK$$

But by (8), $M^{-1}=\frac{1}{N(M)}\overline{M}$. Thus we have

$$-KMK=\frac{1}{N(M)}\overline{M}.$$

This is the same condition as (13). Thus $M=AK$ where A is a K-skew-symmetric, K-orthogonal matrix.

Conversely, if $M=AK$, where A is K-skew-symmetric and K-orthogonal, by definition M is skew-symmetric and since $A'KA=K$ we have $M'KM=K$.

6. Conclusion

This paper, together with Pearl's [3], completes the proof of Riesz' Theorem on all real, four-dimensional vector spaces, regardless of the metric imposed on the space.

Unfortunately, the methods of this paper are rather restricted, and it is not expected that they may be readily applied in the consideration of spaces of higher dimension.

7. References

[1] Dickson, L. E., Algebras and Their Arithmetics (Chicago, 1923).

[2] Katz, L., Olkin, I., Properties and factorizations of matrices defined by the operation of pseudo-transposition, Duke Math. J. **20,** 331–337 (1953).

[3] Pearl, M., On a Theorem of M. Riesz, J. of Res. NBS **62,** No. 3, 89–94 (1959) RP 2935.

[4] Riesz, M., Clifford Numbers and the Equation of Dirac, (unpublished lectures, University of Maryland, 1957–58).

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This paper describes recent progress at the National Bureau of Standards in the field of combinatorial optimization.

Other NBS Publications

J. Res. NBS 70A (Phys. and Chem.), No. 6 (Nov.–Dec. 1966), $1.00.

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J. Res. NBS 71A (Phys. and Chem.), No. 1 (Jan.–Feb. 1967), $1.00.

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Standard Reference Materials: Determination of oxygen in ferrous materials SRM 1090, 1091, and 1092, O. Menis and J. T. Sterling, Misc. Publ. 260–14, (Sept. 23, 1966), 30 cents.

Hydraulic research in the United States 1966, ed. H. K. Middleton and G. Kulin, Misc. Publ. 280, (Sept. 8, 1966), $1.50.

High temperature properties and decomposition of inorganic salts. Part 1. Sulfates, K. H. Stern and E. L. Weise, NSRDS–NBS7, (Oct. 1, 1966), 35 cents.

Mat-formed wood particleboard, CS236–66, (Apr. 15, 1966), 10 cents.

A survey of some empirical and semi-empirical interatomic and intermolecular potentials, B. M. Axilrod, Tech. Note 246, (Oct. 3, 1966), 40 cents.

Research on crystal growth and characterization at the National Bureau of Standards June 1966, ed. H. C. Allen, Jr., Tech. Note 293, (Aug. 26, 1966), 25 cents.

Disclosures on: A transrotor engine, high temperature platinum resistance thermometer, dynamic analog correlation system, and combination metering and safety valve for filling sonde balloons with hydrogen, ed. D. Robbins and A. J. Englert, Tech. Note 295, (Oct. 21, 1966), 25 cents.

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Microchemical Analysis Section: summary of activities July 1965 to June 1966, ed. J. K. Taylor, Tech. Note 403 (Sept. 1, 1966), 50 cents.

Organic chemistry: air pollution studies; kinetic behavior of sugars in solution; carbon-14- and tritium-labeled carbohydrates; characterization of chemical structures, phenylhydrazono-phenylazo tautomerism: synthesis of research materials, cyclopentitols and related substances; novel research materials; standard reference materials (organic) July 1965 through June 1966, ed. H. S. Isbell, Tech. Note 405 (Sept. 30, 1966), 60 cents.

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Improved Fortran program for single particle energy levels and wave functions in nuclear structure calculations, R. S. Caswell, Tech. Note 410 (Sept. 30, 1966), 25 cents.

Analysis of lanolin in soap, F. J. Linnig, Soap Chem. Specialties **XVLL**, No. 7, 55–57 (July 1966).

Analytical applications of microwave spectroscopy, D. R. Lide, Jr., Book, Advances in Analytical Chemistry and Instrumentation **5**, 235–277 (Interscience Publ., New York, N.Y., 1966).

Chemical substructure searching with linear notations, B. A. Marron, G. R. Boltsky, and S. J. Tauber, J. Chem. Doc. **6,** 92–95 (May 1966).

Determination of glycerol in paper, E. L. Graminski and B. W. Forshee, Tappi **49,** No. 7, 324–326 (July 1966).

Electron spin resonance studies of the reaction of water-soluble polymers with hydroxyl radicals, F. Sicilio, M. Dousset, R. E. Florin, and L. A. Wall (Proc. 150th American Chemical Society Meeting, Atlantic City, N.J., Sept. 12–17, 1965), Polymer Preprint **6,** No. 2, 958–964 (Sept. 1965).

Experimental and theoretical studies of photoionization efficiency curves for C_2H_2 and C_2D_2, R. Botter, V. H. Dibeler, J. A. Walker, and H. M. Rosenstock, J. Chem. Phys. **44,** No. 3, 1271–1278 (Feb. 1966).

Gas-phase radiolysis and photolysis of neopentane, S. G. Lias and P. Ausloos, J. Chem. Phys. **43,** No. 8, 2748–2759 (Oct. 1965).

Identification of inclusions with the electron probe microanalyzer, K. F. J. Heinrich, Am. Soc. Testing Mater. Spec. Tech. Publ. **393** (Jan. 1966).

Intramolecular insertion of isobutylidene in the vacuum ultraviolet photolysis of isobutane, E. Tschuikow-Roux and J. R. McNesby, Trans. Faraday Soc. **62,** No. 524, 2158–2163 (Aug. 1966).

Investigation of polymer pyrolysis with thermogravimetry, J. H. Flynn and L. A. Wall (Proc. 150th American Chemical Society Meeting, Atlantic City, N.J., Sept. 12–17, 1965), Polymer Preprint **6,** No. 2, 945–955 (Sept. 1965).

Mass spectrometric study of the reactions of O atoms with NO and NO_2, Erratum, J. T. Herron, J. Chem. Phys. **44,** No. 9, 3645 (May 1966).

Oxidation of asphalt flux with oxides of nitrogen, P. G. Campbell and J. R. Wright, Am. Chem. Soc. Div. Petroleum Chem. **10,** No. 1, 79–89 (Mar. 1965).

Phenylhydrazono-phenylazo tautomerism. Part I. *xylo*-4,5,6-trihydroxy-2-oxo-1,3-bis(phenylhydrazono) cyclohexane and 4-*oxo*-1-phenyl-5-phenylazo-3-pyridazine derivatives, H. S. Isbell and A. J. Fatiadi, Carbohydrate Res. **2,** No. 2, 204–215 (June 1966).

Polymerization and pyrolysis of poly-1,2-dihydronaphthalene, L. A. Wall, L. J. Fetters, and S. Straus (Proc. 150th American Chemical Society Meeting, Atlantic City, N.J., Sept. 12–17, 1965), Polymer Preprint **6,** No. 2, 930–939 (Sept. 1965).

Radiation-induced polymerization at high pressure of 3,3,3-trifluoropropene: 4,4,4,3,3-pentafluorobutene-1; and 5,5,5,4,4,3,3-heptafluoropentene-1, D. W. Brown (Proc. 150th American Chemical Society Meeting, Atlantic City, N.J., Sept. 12–17, 1965), Polymer Preprint **6,** No. 2, 965–976 (Sept. 1965).

Solid-vapor equilibrium in the system helium-methane, M. J. Hiza and A. J. Kidnay (Proc. 1965 Cryogenic Engineering Conf., Rice University, Houston, Tex., Aug 23–25, 1965), Book, Advances in Cryogenic Engineering 11, 338–348 (Plenum Press Inc., New York, N.Y., 1966).

Status of light-element heat-capacity calorimetry at the National Bureau of Standards: a review of the high-temperature thermodynamics of the BeO-H_2O system, T. B. Douglas, Proc. 4th Meeting Interagency Cehmical Rocket Propulsion Group, Working Group on Thermochemistry, Kennedy Space Flight Center, Florida, Mar. 16–17, 1966, I, No. 108, 27 (Chemistry Propulsion Information Agency, Silver Spring, Md., June 1966).

The crystal and molecular structure of dichloro(1,10-phenanthroline) zinc, C. W. Reimann, S. Block, and A. Perloff, Inorg. Chem. **5,** No. 7, 1185–1189 (July 1966).

The crystal structure of the high temperature form of barium borate, $BaO \cdot B_2O_3$, A. D. Mighell, A. Perloff, and S. Block, Acta Cryst. **20,** pt. 6, 819–823 (June 1966).

The determination of bound styrene in soluble high styrene-butadiene resins, M. A. Post, J. Paint Tech. Eng. Official Digest **38,** No. 497, 335–342 (June 1966).

The determination of heats of formation of refractory compounds, G. T. Armstrong and E. S. Domalski, Proc. 4th Meeting Interagency Chemical Rocket Propulsion Group, Working Group on Thermochemistry, Kennedy Space Flight Center, Fla.,March 16–18, 1966, I, No. 108, 15 (Chemistry Propulsion Information Agency, Silver Spring, Md., June 1966).

The determination of mercury in latex paints and paint films containing mercury fungicides, H. W. Berger, J. Paint Technol. **38,** No. 498, 371–376 (July 1966).

The evolution of the accuracy of isotopic analysis by thermal ionization from 2% to 0.2%, W. R. Shields (Proc. Symp. Nuclear Materials Management, IAEA, Vienna, Austria, Aug. 30–Sept. 3, 1965), Book, Nuclear Materials Management, pp. 737–746 (Intern. Atomic Energy Agency, Vienna, Austria, Feb. 1966).

The monomer-polymer equilibrium of alpha-trideuteromethyl-beta, beta-dideuterostyrene, L. J. Fetters and L. A. Wall (Proc. 150th American Chemical Society Meeting, Atlantic City, N.J., Sept. 12–17, 1965), Polymer Preprint **6,** No. 2, 899–903 (Sept. 1965).

The radiation induced polymerization of hexafluoropropylene at high temperature and pressure, R. E. Lowry, D. W. Brown, and L. A. Wall (Proc. 150th American Chemical Society Meeting, Atlantic City, N.J., Sept. 12–17, 1965), Polymer Preprint **6,** No. 2, 977–984 (Sept. 1965).

Thermal decomposition of hexamethylethane, 2,2,3-trimethylbutane and neopentane in a single pulse shock tube, W. Tsang, J. Chem. Phys. **44,** No. 11, 4283–4295 (June 1, 1966).

Thermolytic chain scission of linear polyethylene and an ethylene-propylene copolymer, H. Yu and L. A. Wall (Proc. 150th American Chemical Society Meeting, Atlantic City, N.J., Sept. 12–17, 1965), Polymer Preprint **6,** No. 2, 940–944 (Sept. 1965).

Vacuum-ultraviolet photolysis of paraffin hydrocarbons, J. R. McNesby, Book, Chemical and Biological Actions of Radiation, **IX,** ch. 2, 39–67 (Masson and Co., Editors and Publ., Paris, France, 1966).

Advances in cryogenic engineering, Book, K. D. Timmerhaus, editor, Proc. 1964 Cryogenic Engineering Conf., University of Pennsylvania, Philadelphia, Pa., Aug. 18–21, 1964, 10, Pt. 1 (Plenum Press, Inc., New York, N.Y., 1965).

Apparatus for studying the effects of atmospheric pollution and cyclic dew formation on the deterioration of materials, J. W. Pitts and D. G. Moore, Mater. Res. Std. **6,** No. 7, 328–333 (July 1966).

A cold moderator refrigerator incorporating a high-speed turbine expander, R. O. Voth, M. T. Norton, and W. A. Wilson (Proc. 1965 Cryogenic Engineering Conf., Rice University, Houston, Tex., Aug. 23–25, 1965), Book, Advances in Cryogenic Engineering 11, 126–138 (Plenum Press Inc., New York, N.Y., 1966).

A new near-zone electric field-strength meter, F. M. Greene (Proc. 8th IEEE Symp. Electromagnetic Compatibility, San Francisco, Calif., July 11–13, 1966), Symposium Digest New Ideas for Electromagnetic Compatibility, 1966.

A proposal for a supplement to the patent system, S. Henig, Design News Mag. **21,** No. 1, 84–89 (Jan. 5, 1966).

Bond with reinforcing steel, D. Watstein, ASTM Significance of Tests of Concrete and Concrete Making Materials, Am. Soc. Testing Mater. Spec. Tech. Publ. **169A,** pp. 239–245 (July 1966).

Chemical admixtures, B. E. Foster, ASTM Significance of Tests of Concrete and Concrete Making Materials, Am. Soc. Testing Mater. Spec. Tech. Publ. **169A,** pp. 556–564 (July 1966).

Comparative study of forced convection boiling heat transfer correlations for cryogenic fluids, P. J. Giarratano and R. V. Smith (Proc. 1965 Cryogenic Eng. Conf., Rice University, Houston, Tex., Aug. 23–25, 1965), Book, Advances in Cryogenic Engineering 11, 492–506 (Plenum Press Inc., New York, N.Y., 1966).

Floor coverings, T. H. Boone, Book, Building Construction Handbook, 2d edition, ed. F. S. Merritt, Sec. 13, pp. 13–1–13–8 (1965).

Free radical microwave absorption meter, H. E. Radford, Rev. Sci. Instr. **37,** No. 6, 790–792 (June 1966).

Instrumentation and measurements, H. E. Bussey and J. E. Gray, Digest of Literature on Dielectrics **28,** ch. I., 1–31 (1964): Natl. Acad. Sci.–Natl. Res. Council Publ. 1324 (Natl. Acad. Sci.–Natl. Res. Council, Washington, D.C., 1965).

International advances in cryogenic engineering, Book, K. D. Timmerhaus, editor, Proc. 1964 Cryogenic Engineering Conf., Univ. Pennsylvania, Philadelphia, Pa., Aug. 18–21, 1964, 10, Pt. 2 (Plenum Press, Inc., New York, N.Y., 1965).

Liquid-solid mixtures of hydrogen near the triple point, D. B. Mann, P. R. Ludtke, C. F. Sindt, and D. B. Chelton (Proc. 1965 Cryogenic Engineering Conf., Rice University, Houston, Tex., 23–25, 1966), Book, Advances in Cryogenic Engineering 11, 207–217 (Plenum Press Inc., New York, N.Y., 1966).

Message from outgoing editor, R. W. Beatty, IEEE Trans. Microwave Theory Tech. **MTT-14,** No. 4, 170 (Apr. 1966).

Metals for use in orthopedics, S. J. Rosenberg, Book, Orthotics Etcetra, ed. S. Licht, 9, 77–94 (E. Licht Publ., New Haven, Conn., 1966).

New standard for the calibration of microphones, W. Koidan, Mag. Std. **37,** No. 5, 141–144 (May 1966).

Resistance to weathering, H. T. Arni, ASTM Significance of Tests of

Concrete and Concrete Making Materials, Am. Soc. Testing Mater. Spec. Tech. Publ. **169A,** pp. 261–274 (1966).

Survey of infrared measurement techniques and computational methods in radiant heat transfer, S. T. Dunn, J. C. Richmond, and J. F. Parmer, J. Spacecraft Rockets **3,** No. 7, 961–975 (July 1966).

Voluntary product standards—what they mean to the consumer, M. Lonie, J. Home Econ. **58,** No. 1, 22 (Jan. 1966).

On the complex structure of the universe, E. H. Brown, J. Math. Phys. **7,** No. 3, 417–425 (Mar. 1966).

Atomic timekeeping and the statistics of precision signal generators, J. A. Barnes, Proc. IEEE **54,** No. 2, 207–220 (Feb. 1966).

A superconducting liquid-level sensor for slush hydrogen use, B. L. Knight, K. D. Timmerhaus, and T. M. Flynn (Proc. 1965 Cryogenic Engineering Conf., Rice University, Houston, Tex., Aug. 23–25, 1965), Book, Advances in Cryogenic Engineering 11, 218–222 (Plenum Press Inc., New York, N.Y., 1966).

Effects of long-term stability on the definition and measurement of short-term stability, J. A. Barnes and D. W. Allan, Proc. IEEE–NASA Symp. Definition and Measurement of Short-Term Frequency Stability, NASA Spec. Publ. No. 80, pp. 119–123 (Nov. 23–24, 1964).

Evaluation of a thallium atomic beam frequency standard at the National Bureau of Standards, R. E. Beehler and D. J. Glaze, IEEE Trans. Instr. Meas. **IM-15,** No. 1 & 2, 55–58 (Mar.–June 1966).

LF–VLF frequency and time services of the National Bureau of Standards, D. H. Andrews, IEEE Trans. Instr. Meas. **IM-14,** No. 4, 233–237 (Dec. 1965).

Optical methods of temperature measurement, G. A. Hornbeck, J. Appl. Opt. Lead Article **5,** No. 2, 179 (Feb. 1966).

Quantity control—A tool of industry, and arm of government, M. W. Jensen, Proc. Annual Conf. Society for the Advancement of Food Service Research, Washington, D.C. (Apr. 18, 1966).

Remote positioner for rotary switches, S. B. Lang, and A. M. Gray, Rev. Sci. Instr. **37,** No. 6, 799–801 (June 1966).

The performance and capability of cesium beam frequency standards at the National Bureau of Standards, R. E. Beehler and D. J. Glaze, IEEE Trans. Instr. Meas. **IM-15,** No. 1 & 2, 48–55 (Mar.–June 1966).

Time resolved microwave interferometry on diagnostic tool for decaying plasma afterglows, A. J. Estin and M. M. Anderson, Rev. Sci. Instr. **37,** No. 4, 468–470 (Apr. 1966).

Analyzing liquid H_2 with NMR, C. E. Miller, W. J. Alspach, and T. M. Flynn, Cryogenic Engr. News **1,** No. 8, 66 (June 1966).

Angular dependence of the one-phonon-nuclear quadrupole interactions, R. J. Mahler, J. Phys. Chem. Solids **27,** 871–879 (1966).

A suggested experiment to measure part of the transverse electromagnetic mass of the electron, E. A. Power, Proc. Roy. Soc. (London) **292,** No. 1430, 424–432 (May 31, 1966).

Brillouin scattering in liquids at 4880 Å, S. L. Shapiro, M. McClintock, D. A. Jennings, and R. L. Barger, IEEE Quantum Electronics **QE-2,** No. 5, 89–93 (May 1966).

Calculated line strengths for the transition array for $d^2s - d^2p$: application to Ni II, H. Mendlowitz, Astrophys. J. **143,** No. 2, 573–590 (Feb. 1966).

Calculated slant-path absorption and distribution of atmospheric water vapor, R. F. Calfee and D. M. Gates, Appl. Opt. **5,** No. 2, 287–292 (Feb. 1966).

Comments on the mechanism of the 377-micron CN laser, H. P. Broida, K. M. Evenson, and T. T. Kikuchi, J. Appl. Phys. **36,** No. 10, 3355 (Oct. 1965).

Comparison of the Lennard-Jones, Exp: 6, and Kihara potential functions from viscosity data of dilute argon, H. J. M. Halney, J. Chem. Phys. **44,** No. 11, 4219–4222 (June 1966).

Direct ratio readings from a universal ratio set, D. Ramaley and J. F. Schafer, Instr. Control Systems **39,** 73–74 (Jan. 1966).

Dielectric relaxation in molecular crystals: multiple site models, J. D. Hoffman, Book, Molecular Relaxation Processes, Chemical Society Spec. Publ. **20,** pp. 47–60 (The Chemical Society and Academic Press, New York, N.Y., 1966).

Electric quadrupole transition in the $A^1\Pi \leftarrow X^1\Sigma^+$ system of CO, J. D. Simmons and S. G. Tilford, J. Chem. Phys. **44,** No. 11, 4145–4147 (June 1966).

Electron impact ionization cross section data for atoms, atomic ions, and diatomic molecules: I. Experimental data, L. J. Kieffer and G. H. Dunn, Rev. Mod. Phys. **38,** No. 1, 1–33 (Jan. 1966).

Franck-Condon factors for the ionization of H_2 and D_2, G. H. Dunn, J. Chem. Phys. **44,** No. 7, 2592–2594 (Apr. 1966).

Fundamentals of ultrasonics, M. Greenspan, J. Acoust. Soc. Am. **37,** No. 1, 158–159 (July 1965).

Impact broadening of microwave spectra, A. Ben-Reuven, Phys. Rev. **145,** No. 1, 7–22 (May 1966).

Infrared spectra of HCl, DCl, HBr, and DBr in solid rare-gas matrices, D. E. Mann, N. Acquista, and D. White, J. Chem. Phys. **44,** No. 9, 3453–3467 (May 1966).

Infrasonic waves from aurorae, K. Maeda and J. M. Young, Nature **207,** No. 4994, 279–281 (July 1965).

Ion mobilities in helium, E. C. Beaty, J. C. Browne, and A. Dalgarno, Phys. Rev. Letters **16,** No. 17, 723–724 (Apr. 1966).

Low-temperature phase transformations, R. P. Reed and J. F. Breedis, Behavior of Materials at Cryogenic Temperatures, Am. Soc. Testing Mater. Spec. Tech. Publ. No. 387, pp. 60–132 (1966).

Low temperature speed of sound in single crystal ice, T. M. Proctor, Jr., J. Acoust. Soc. Am. **39,** No. 5, 972–977 (May 1966).

Magnetic core permeability measurement techniques, R. D. Harrington and A. L. Rasmussen, Proc. Magnetic Core Conf. **7,** 11–24 (1965).

Measurements of collisional energy transfer between rotational energy levels in CN, K. M. Evenson and H. P. Broida, J. Chem. Phys. **44,** No. 4, 1637–1641 (Feb. 15, 1966).

Measurements of the viscosity of parahydrogen, D. E. Diller (Proc. IXth Intern. Conf. Low Temperature Physics, Columbus, Ohio, Aug. 31–Sept. 4, 1964), Book, Low Temperature Physics, LT9, Pt. B, pp. 1227–1229 (Plenum Press Inc., New York, N.Y., 1965).

Nonlinear diffusion with recombination in an electron beam excited plasma, E. R. Mosburg, Jr., Phys. Fluids **9,** No. 4, 824–826 (Apr. 1966).

Nuclear spin relaxation in liquid hydrogen, C. E. Miller, T. M. Flynn, T. K. Grady, and J. S. Waugh, Physica **32,** No. 2, 244–251 (Feb. 1966).

One dimensional solution of the Ginzburg-Landau equations for thin superconducting films, V. D. Arp, R. S. Collier, R. A. Kamper, and H. Meissner, Phys. Rev. **145,** No. 1, 231–236 (May 6, 1966).

Physics of the *D* region at high latitudes, G. C. Reid (Proc. NATO Advanced Study Institute, Finse, Norway, April 1965), Book, Electron Density Profiles in Ionosphere and Exosphere, ed. J. Frihagen, pp. 17–26 (North Holland Publ. Co., Amsterdam, The Netherlands, 1966).

Polycrystalline spin wave theory of ferromagnetic resonance compared with the tilting experiment, A. S. Risley, E. G. Johnson, Jr., and H. E. Bussey, J. Appl. Phys. **37,** No. 2, 656–668 (Feb. 1966).

Radiation lifetime of the first $2_{p3/2}$ state of ionized calcium and magnesium by the Hanle effect, W. W. Smith and A. C. Gallagher, Phys. Rev. **145,** No. 1, 26–35 (May 6, 1966).

Saturation effects in the sensitized fluorescence of CaF_2(Ce, Mn), B. L. Danielson, Phys. Rev. **142,** No. 1, 228–230 (Feb. 4, 1966).

Soft x-ray spectrum of Ni and comparison with photoemission and ion neutralization results, J. R. Cuthill, A. J. McAlister, and M. L. Williams, Phys. Rev. Letters **16,** No. 22, 993–995 (May 30, 1966).

Some properties of the long-time values of the probability densities for moderately dense gases, R. A. Piccirelli, J. Math. Phys. **7,** 922 (1966).

Summary of recent determinations of the ultrasonic velocity in fluid parahydrogen, B. A. Younglove (Proc. IXth Intern. Conf. Low Temperature Physics, Columbus, Ohio, Aug. 31–Sept. 4, 1964), Book, Low Temperature Physics **LT9,** pt. B., pp. 1223–1226 (Plenum Press Inc., New York, N.Y., 1965).

The Chapman-Enskog solution of the generalized Boltzmann equation, M. S. Green, L. S. Garcia-Colin, and F. Chaos, Physica **32,** 450 (1966).

The flow field of a body-stabilized two-dimensional V-flame, F. W. Ruegg and W. W. Dorsey, Combustion and Flame **10,** No. 1, 1–10 (Dec. 1966).

The role of hydrogen in the stress-corrosion cracking of low carbon steel in a nitrate solution, H. L. Logan and H. T. Yolken, Proc. 2d Intern. Congress Metallic Corrosion, New York, N.Y., March 1963, p. 109 (Natl. Assoc. Corrosion Engr., Houston, Texas, 1966).

The spheroidal method in the theory of the orbit of an artificial satellite, J. P. Vinti, Proc. Symp. Celestial Mechanics, Mathematische Forschungsinstitut, Oberwolfach, Germany, March 1964, pp. 97–111 (1964).

The theory of chromospheric spectrum, J. T. Jefferies, Book, The Solar Spectrum, ed. C. DeJager, pp. 131-150 (D. Reidel Publ. Co., Dordrecht, The Netherlands, 1965).

Thermodynamics of the rigid-rotor at high temperature, J. E. Kil-

patrick, Y. Fukuda, and S. Y. Larsen, J. Chem. Phys. **43,** No. 2, 430–432 (July 1965).

Thermophysical properties of metals at cryogenic temperatures, R. L. Powell, Behavior of Materials at Cryogenic Temperatures, Am. Soc. Testing Mater. Spec. Tech. Publ. No. 387, pp. 134–148 (1966).

Vacuum ultraviolet spectrum of neutral silicon, V. Kaufman, L. J. Radziemski, Jr., and K. L. Andres, J. Opt. Soc. Am. **56,** No. 7, 911–915 (July 1966).

Zero-point energy and Lamb shift, E. A. Power, Am. J. Phys. **34,** No. 6, 516–518 (June 1966).

Computed performance of moderate size, super-gain, end-fire antenna arrays, C. O. Stearns, IEEE Trans. Ant. Prop. **AP-14,** No. 2, 241–242 (Mar. 1966).

Electron precipitation and ionospheric radio absorption in the auroral zones, K. L. Hargreaves and R. D. Sharp, Planetary Space Sci. **13,** 1171–1183 (Dec. 1965).

Evidence of long-period acoustic-gravity waves launched into the *F* region by the Alaskan earthquake of March 28, 1964, R. V. Row, J. Geophys. Res. Letter **71,** 343–345 (Jan. 1966).

International URSIGRAM and world days service "report on activities in 1964," A. H. Shapley, URSI Info. Bull. No. 150, pp. 43–55 (May 1965).

Ionospheric effects of particles, G. C. Reid, Proc. COSPAR Polar Cap Panel Symp., Alpach, Austria, 1964, pp. 221–231 (Academic Press Inc., New York, N.Y., 1965).

Linear tuning of a microwave cavity, M. J. Vetter, IEEE Trans. Microwave Theory Tech. **MTT-13,** No. 6, 880 (Nov. 1965).

Observations of ionospheric irregularities and plasma resonances by the fixed-frequency topside sounder satellite, W. Calvert (Proc. NATO Advanced Study Institute, Finse, Norway, April 1965), Book, Electron Density Profiles in Ionosphere and Exosphere, ed. J. Frihagen, pp. 281-298 (North Holland Publ. Co., Amsterdam, The Netherlands, 1966).

Observations of the equatorial ionosphere using incoherent backscatter, D. T. Farley (Proc. NATO Advanced Study Institute, Finse, Norway, April 1965), Book, Electron Density Profiles in Ionosphere and Exosphere, ed. J. Frihagen, pp. 446–469 (North Holland Publ. Co., Amsterdam, The Netherlands, 1966).

Slow fluctuations between conjugate points in the auroral absorption of cosmic noise, H. J. A. Chivers and J. K. Hargreaves, J. Atmos. Terrest. Phys. **28,** 337–342 (1966).

SPACEWARN—an international mechanism for rapid distribution of information on satellites and space probes, A. H. Shapley, COSPAR Info. Bull. No. 24, pp. 30–55 (July 1965).

Spectral distribution of solar radiation at the earth's surface, D. M. Gates, Science **151,** No. 3710, 523–529 (Feb. 1966).

Statistics of atomic frequency standards, D. W. Allan, Proc. IEEE **54,** No. 2, 221–230 (Feb. 1966).

Swept-frequency antenna gain measurements, R. G. FitzGerrell, IEEE Trans. Ant. Prop. **AP-14,** No. 2, 173–178 (Mar. 1966).

The important atmospheric processes in the *D* region, G. C. Reid, Proc. COSPAR Polar Cap Panel, Alpach, Austria, 1964, pp. 15–19 (Academic Press Inc., New York, N.Y., 1965).

"Winter anomaly" in ionospheric absorption and stratospheric warnings, A. H. Shapley and W. J. G. Beynon, Nature **206,** 1242–1243 (June 1965).

Dissociation of acetic acid-d_4 in deuterium oxide from 5 to 50° and related isotope effects, M. Paabo, R. G. Bates, and R. A. Robinson, J. Phys. Chem. **70,** 2073–2077 (1966).

Effects of errors in the chemical literature on the compilation of critically evaluated data, W. H. Evans, J. Chem. Doc. **6,** No. 3, 135–136 (Aug. 1966).

Electronically excited NO by photodissociation of NO_2 and NOCl, K. H. Welge, J. Chem. Phys. **45,** No. 4, 1113–1117 (Aug. 15, 1966).

Gas-phase photolysis of isopentane at photon energies below and above the ionization energy, A. A. Scala and P. Ausloos, J. Chem. Phys. **45,** No. 3, 847–854 (Aug. 1, 1966).

Formation of $N_2(A^3\Sigma^+)$ and $N(^2D, ^2P)$ by photodissociation of HN_3 and N_2O and their reactions with NO and N_2O, K. H. Welge, J. Chem. Phys. **45,** No. 1, 166–170 (July 1, 1966).

Glass-molten salt interactions, K. H. Stern, Chem. Rev. **66,** No. 4, 355–372 (July 25, 1966).

Ground state of bis-(acetylacetonato) copper (II), H. C. Allen, Jr., J. Chem. Phys. **45,** No. 2, 553–555 (July 15, 1966).

Interactions in aqueous nonelectrolyte solutions. I. Solute-solvent equilibria, R. H. Stokes and R. A. Robinson, J. Phys. Chem. **70,** 2126–2131 (1966).

K-absorption fine structures of sulfur in gaseous SF_6, R. E. LaVilla and R. D. Deslattes, J. Chem. Phys. **44,** No. 12, 4399–4400 (June 15, 1966).

Losses due to adsorption during filtration of aqueous solutions of polycyclic aromatic hydrocarbons, M. Inscoe, Nature **211,** No. 5053, 1083–1085 (Sept. 3, 1966).

Mass spectrometric study of the rates of the reactions of nitrogen atoms with olefins, J. T. Herron, Phys. Chem. **70,** No. 9, 2803–2807 (Sept. 1966).

Mass-spectrometric study of photoionization. IV. Ethylene and 1,2-dideuteroethylene, R. Botter, V. H. Dibeler, J. A. Walker, and H. M. Rosenstock, J. Chem. Phys. **45,** No. 4, 1298–1301 (Aug. 15, 1966).

Measurement of photochemical degradation in rigid poly(vinly chloride) by colour reactions with *N,N*-dimethyl-*p*-phenylenediamine, J. R. Wright and V. E. Gray (Proc. Conf. Plastics in Building Structures, London, England, June 14–16, 1965), Plastics Inst. Trans. J. Suppl. **1,** 113–118 (June 14, 1966).

Mossbauer spectrometry, J. R. DeVoe and J. J. Spijkerman, Anal. Chem. **38,** 382R–393R (1966).

Photoionization efficiency curve for SF_6 in the wavelength region 1050 to 600 Å, V. H. Dibeler and J. A. Walker, J. Chem. Phys. **44,** No. 12, 4405–4406 (June 15, 1966).

Photoinoization of *n*-butane at 1067–1048 Å. Decomposition of the parent ion and superexcited molecule, P. Ausloos and S. G. Lias, J. Chem. Phys. **45,** No. 2, 524–535 (July 15, 1966).

Study of D_2O ice as a cold-neutron source, J. J. Rush, D. W. Connor, and R. S. Carter, Nucl. Sci. Engr. **25,** 383–389 (1966).

An examination of the effects of heat transfer and compressible flow on the performance of laminar flowmeters, F. W. Ruegg and H. H. Allion, Proc. Flow Measurement Conf., Am. Soc. Mech. Eng. Fluid Meters Golden Anniversary, Pittsburgh, Pa., Sept. 26–28, 1966, pp. 253–273 (1966).

A high directivity, broadband coaxial coupler, P. A. Hudson, IEEE Trans. Microwave Theory Tech. **MTT-14,** No. 6, 293–294 (June 1966).

Development of performance standards, B. E. Foster, Military Engr. **58,** No. 385, 335–336 (Sept.–Oct. 1966).

Roofing research, T. H. Boone, Military Engr. **385,** 344–345 (Sept.–Oct. 1966).

Temperature effects on pressure transducers, J. W. Dean and T. M. Flynn, ISA Trans. **5,** No. 3, 223–232 (July 1966).

Analysis of multimode experimental data, H. Kurss, IEEE Trans. Ant. Prop. **AP-14,** No. 3, 401–402 (May 1966).

A network-simulation approach to the railroad freight train scheduling and car sorting problem, W. P. Allman, Proc. 4th Intern. Conf. Operational Research, Boston, Mass., Aug. 29–Sept. 2, 1966 (Arthur D. Little Inc., Cambridge, Mass., 1966).

A new guide for photographers, C. S. McCamy, Mag. Stds. **37,** No. 8, 223–224 (Aug. 1966).

Absorption spectrum of SF_6 in the far ultraviolet by electron impact, J. A. Simpson, C. E. Kuyatt, and S. R. Mielczarek, J. Chem. Phys. **44,** No. 12, 4403–4404 (June 15, 1966).

A simple non-scanning camera for x-ray diffraction contrast topography, H. P. Layer and R. D. Deslattes, J. Appl. Phys. **37,** No. 9, 3631–3632 (Aug. 1966).

Chemical and physical mechanisms of salt stress-corrosion cracking in the titanium 8-1-1 alloy, H. L. Logan, M. J. McBee, C. J. Bechtoldt, B. T. Sanderson, and G. M. Ugiansky (Proc. Stress Corrosion Cracking Symp., Seattle, Washington, Nov. 1965), Am. Soc. Testing Material Spec. Tech. Publ. **397,** pp. 215–229 (1966).

Commission on fundamental spectroscopic data, C. Moore-Sitterly, Trans. Intern. Astron. Union **XIIB,** 173–185 (1966).

Current algebras and matrix elements of the axivector current, H. J. Lipkin, H. R. Rubinstein, and S. Meshkov, Phys. Rev. **148,** No. 4, 1405–1407 (Aug. 26, 1966).

Electron microscopy of irradiation-induced defect clusters in magnesium fluoride, D. J. Barber, Phys. Stat. Sol. **16,** 531–547 (1966).

Exchange and direct second virial coefficients for hard spheres, M. E. Boyd, S. Y. Larsen, and J. E. Kilpatrick, J. Chem. Phys. **45,** No. 2, 499–508 (July 15, 1966).

Fracture surface energy of soda-lime glass, S. M. Wiederhorn (Proc. Conf. Role of Grain Boundaries and Surface in Ceramics, North Carolina State University, Raleigh, N.C., Nov. 16–18, 1964), Book,

Materials Science Research **3,** 503–528 (Plenum Press Inc., New York, N.Y., 1966).

Heat transfer to a subliming solid-vapor mixture of hydrogen below its triple point, M. C. Jones, T. T. Magamoto, and J. A. Brennan, A.I.Ch.E. J. **12,** No. 4, 790–795 (July 1966).

Higher symmetries and the 2^+ mesons, D. Horn, J. J. Coyne, S. Meshkov, and J. C. Carter, Phys. Rev. **147,** No. 4, 980–984 (July 29, 1966).

Ionization energies of the neutral rare earths, J. Reader and J. Sugar, J. Opt. Soc. Am. **56,** No. 9, 1189–1194 (Sept. 1966).

L-uncoupling effects on the electron-paramagnetic-resonance spectra of $N^{14}O^{16}$, R. L. Brown and H. E. Radford, Phys. Rev. **147,** No. 1, 6–12 (July 8, 1966).

Magnetization and critical fields of superconducting $SrTiO_3$, E. Ambler, J. H. Colwell, W. R. Hosler, and J. F. Schooley, Phys. Rev. **148,** No. 1, 280–286 (Aug. 5, 1966).

Melting temperatures of copolymers, J. P. Colson and R. K. Eby, J. Appl. Phys. **37,** No. 9, 3511–3514 (Sept. 1966).

Quench defects formed in a low-stacking-fault-energy silver-tin alloy, A. W. Ruff, Jr. and L. K. Ives, J. Appl. Phys. **37,** No. 8, 3073–3079 (July 1966).

Radiation absorption between the ultraviolet and x-ray bands, U. Fano, Science **153,** No. 3735, 522–525 (July 29, 1966).

Structure in the photoionization continuum of SF_6 below 630Å, K. Codling, J. Chem. Phys. **44,** No. 12, 4401–4402 (June 15, 1966).

Studies of some exploding wire light sources, E. C. Cassidy and S. Abramowitz, J. SMPTE **75,** No. 8, 735–737 (Aug. 1966).

SU_3 reaction predictions and symmetry breaking, S. Meshkov, Elementary Particles Conf., Institute for Theoretical Physics, University of Colorado, Boulder, Colo., Summer 1964, Lectures Theoret. Phys. **VII B,** 36–50 (1965).

Triple collision contribution to the transport coefficients of a rigid sphere gas, J. V. Sengers, Phys. Fluids **9,** No. 7, 1333–1347 (July 1966).

Viscoelastic behavior, H. Leaderman, Encyclopaedic Dictionary of Physics **7,** 633–634 (Pergamon Press Inc., London, England, 1962).

Medium-frequency sky wave propagation in middle and low latitudes, A. F. Barghausen, IEEE Trans. Broadcasting **BC–12,** No. 1, 1–14 (June 1966).

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